Sobre la no existència de petits breathers en equacions de Klein-Gordon

T. M-Seara

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Collaborators: O. Gomide, M. Guardia, Ch. Zeng

Breathers and the Klein-Gordon equation

- Klein-Gordon equation

$$u_{tt} = u_{xx} - u + g(u), \qquad g(0) = g'(0) = 0, \qquad x \in \mathbb{R}$$

- Breathers: Periodic in t localized in x solutions $u(x,t)$.
- The linearized Klein-Gordon equation

$$u_{tt} = u_{xx} - u, \qquad x \in \mathbb{R}$$

  has linear decay as $t \to \infty$
- The existence of Breathers shows a big non-linear effect: Breathers are an "obstacle" to non-linear decay
- Breathers for the Sine-Gordon equation $u_{tt} = u_{xx} - \sin u$:

$$u(x,t) = 4 \arctan \left(\frac{m}{\omega} \frac{\sin(\omega t)}{\cosh(mx)} \right), \quad m, \omega > 0, \; m^2 + \omega^2 = 1.$$

  They are $\frac{2\pi}{\omega}$-periodic in time and $\lim_{x \to \pm\infty} u(x,t) = 0$.
- What about other nonlinearities?
- Families of breathers should be unlikely to happen.

Spatial dynamics: Breathers as homoclinic orbits

$$u_{tt} = u_{xx} - u + g(u), \qquad g(u) = \mathcal{O}(u^2)$$

- Dynamical system with x as time: phase space is space of $2\pi/\omega$-periodic functions in t for some $\omega > 0$.
- Breathers $\equiv$ Homoclinic orbits to the steady state $u = 0$ in an infinite dimensional phase space.
- $u = 0$ has finite dimensional stable and unstable eigenspaces: the stable/unstable invariant manifolds unlikely will intersect.
- But this is hard to prove in general...
- Breathers do exist for Hamiltonian systems on lattices (McKay, Aubry,...).

Non-existence of breathers for the Klein Gordon eq.

Global results:

- Kowalczyk, Martel and Muñoz (2016): Nonexistence of odd (in x) breathers for any odd g.
- The breathers of the Sine-Gordon equation are even in x!

Perturbative results:

- Birnir–McKean–Weinstein and Denzler (1990's): Perturbed Sine-Gordon equation

$$u_{tt} = u_{xx} - \sin u + \varepsilon \Delta(u), \quad \varepsilon \ll 1, \quad \Delta \text{ analytic}$$

- Persistence of the family of breathers implies $\Delta(u)$ is a trivial perturbation.

Small amplitude breathers for the odd Klein Gordon eq.

- What about (families of) small amplitude breathers?
- Equivalent to small homoclinic loops to $u = 0$.
- Simplest setting: Odd Klein-Gordon equation

$$\partial_t^2 u - \partial_x^2 u + u - \frac{1}{3}u^3 - f(u) = 0 \quad f(u) = \mathcal{O}(u^5), \text{ odd}$$

- Soffer–Weinstein (1999) and Bambusi–Cuccagna (2011): Non-existence of breathers if one adds a potential (under some hypotheses).

Kruskal and Segur

- Kruskal–Segur (1987): Formal arguments for the ϕ^4 model to indicate the breakdown of breathers with

$$\text{frequency } \omega : 0 < 1 - \omega \ll 1 \quad \text{and} \quad \text{amplitude } \sim \sqrt{1-\omega^2}.$$

- Questions:

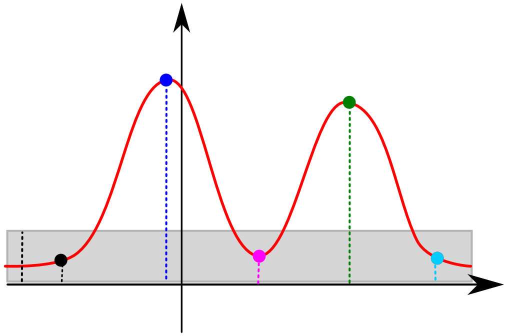
  1. How to make rigorous the formal arguments to prove the breakdown of breathers and extend the proof to all possible ω's.
  2. Do small amplitude breathers with exponentially small (with respect to the amplitude) tails exist? $\leftarrow$ Generalized breathers

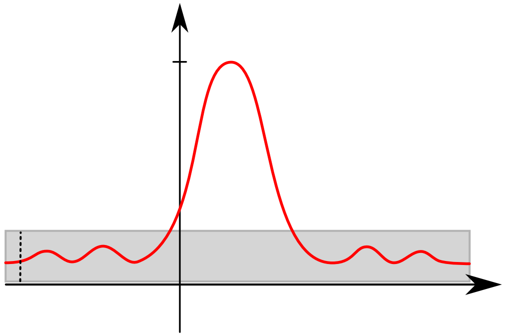
Small amplitude breathers for the odd Klein Gordon eq.

- Goal: For a "typical" analytic odd f, small amplitude breathers do not exist.
- But we need to impose certain restrictions...
- Let $\sigma \in (0,1)$ and $\omega > 0$. A $\frac{2\pi}{\omega}$-periodic-in-t function $u(x,t)$ is σ-multi-bump in x if there exist $x_1 < x_2 < x_3 < x_4 < x_5$ such that

$$\|u(x_j,\cdot)\|_{\ell^1} \leq \sigma \|u(x_i,\cdot)\|_{\ell^1}, \quad \forall j \in \{1,3,5\},\ i \in \{2,4\}.$$

Otherwise, it is said to be σ-single-bump.


Multi-bump


Single-bump

Main result: Non-existence of breathers

$$\partial_t^2 u - \partial_x^2 u + u - \frac{1}{3}u^3 - f(u) = 0, \quad f(u) = \mathcal{O}(u^5), \text{ odd, analytic}$$

Theorem (Guàrdia-Gomide-S.-Zeng)

There exists $\Theta_f \in \mathbb{C}$, depending analytically on f, such that if $\Theta_f \neq 0$: For any $\sigma \in (0,1)$, there exists $\rho^* > 0$ such that there does not exist any solution $u(x,t)$ which:

1. *is $\frac{2\pi}{\omega}$-periodic in t for some $\omega > 0$,*
2. *satisfies*

$$\|u(x,\cdot)\|_{H^1_t\left(\left(-\frac{\pi}{\omega},\frac{\pi}{\omega}\right)\right)} + \|\partial_x u(x,\cdot)\|_{L^2_t\left(\left(-\frac{\pi}{\omega},\frac{\pi}{\omega}\right)\right)} \to 0, \quad \text{as} \quad |x| \to +\infty,$$

3. *satisfies $\sup_{x\in\mathbb{R}} \|u(x,\cdot)\|_{\ell^1} < \min\{1, \rho^* \omega^{\frac{1}{2}}\}$,*
4. *is σ-single-bump.*

Some remarks

- Θ_f depends analytically on $f \to$ For "typical" f, $\Theta_f \neq 0$.
- So, for typical f, small amplitude breathers do not exist provided:
  - We restrict to single-bump breathers,
  - We admit the smallness to depend on ω:

$$\sup_{x\in\mathbb{R}} \|u(x,\cdot)\|_{\ell^1} < \min\{1, \rho^*\omega^{\frac{1}{2}}\}.$$

- With some extra work we should be able to prove that multi-bump breathers should not exist either.
- One should be able to rule out breathers such that

$$\sup_{x\in\mathbb{R}} \|u(x,\cdot)\|_{\ell^1} < \rho^*.$$

Generalized breathers

- Nan Lu (2014): There exist breathers with exponentially small tails for some periods.
- Fix the frequency $\omega = \sqrt{1-\varepsilon^2}$ with $0 < \varepsilon \ll 1$.
- There exist solutions u such that are $2\pi/\sqrt{1-\varepsilon^2}$ – periodic in time and

$$\frac{\varepsilon}{2} \leq \sup \|u(x,\cdot)\|_{H^1_t\left(-\frac{\pi}{\omega},\frac{\pi}{\omega}\right)} \leq 2\varepsilon$$

  and

$$\limsup_{x\to\pm\infty}\{\|u(x,\cdot)\|_{H^1_t\left((-\frac{\pi}{\omega},\frac{\pi}{\omega})\right)} + \|\partial_x u(x,\cdot)\|_{L^2_t\left((-\frac{\pi}{\omega},\frac{\pi}{\omega})\right)}\} \lesssim e^{-c/\varepsilon}, \quad c > 0.$$

- Groves and Schneider (2000's): "modulated pulse" solutions with small (beyond all orders) tails for the nonlinear Klein Gordon equations (and quasilinear wave equations).

Main results: Generalized breathers

Theorem (Guàrdia-Gomide-S.-Zeng)

Fix the frequency $\omega = \sqrt{1-\varepsilon^2}$ *with* $0 < \varepsilon \ll 1$.

- *There exist* $2\pi/\omega$ *-periodic-in-t solutions u such that*

$$\frac{\varepsilon}{2} \leq \sup \|u(x,\cdot)\|_{H_t^1\left(-\frac{\pi}{\omega},\frac{\pi}{\omega}\right)} \leq 2\varepsilon \qquad \text{and}$$

$$\lim_{|x|\to\infty} \sup \left(\|u(x,\cdot)\|_{H_t^1\left(-\frac{\pi}{\omega},\frac{\pi}{\omega}\right)} + \|\partial_x u(x,\cdot)\|_{L_t^2\left(-\frac{\pi}{\omega},\frac{\pi}{\omega}\right)}\right) \leq M e^{-\frac{\sqrt{2}\pi}{\varepsilon}}.$$

- *If* $\Theta_f \neq 0$, *they also satisfy*

$$\lim_{|x|\to\infty} \inf \left(\|u(x,\cdot)\|_{H_t^1\left(-\frac{\pi}{\omega},\frac{\pi}{\omega}\right)} + \|\partial_x u(x,\cdot)\|_{L_t^2\left(-\frac{\pi}{\omega},\frac{\pi}{\omega}\right)}\right) \geq M^{-1} e^{-\frac{\sqrt{2}\pi}{\varepsilon}}.$$

Some ideas about the proof

- The proofs of all the results rely on spatial dynamics techniques (x as evolution variable). Breathers are homoclinic orbits to $u = 0$.
- For the breackdown of breathers: We need to analyze the stable/unstable invariant manifolds associated to the steady state $u = 0$.
- For the generalized breathers: center-stable and center-unstable invariant manifolds.
- In this talk, we focus on the proof of the breakdown of breathers.
- Small breathers correspond to small homoclinics and these appear at bifurcations!
- We need to deal with exponentially small phenomena.

Breather breakdown from spatial dynamics point of view

- Choose any frequency $\omega > 0$ and fix periodicity in t to be $2\pi/\omega$.
- Change time to $\tau = \omega t$, and consider $u(\tau, x)$ 2π periodic in τ satisfying:

$$\omega^2 \partial_{\tau\tau} u - \partial_{xx} u + u - \frac{1}{3} u^3 - f(u) = 0, \quad f(u) = \mathcal{O}(u^5), \text{odd, analytic}$$

which is a (Hamiltonian) equation depending on a parameter ω.

- Linearization around $u = 0$: $\omega^2 \partial_{\tau\tau} u - \partial_{xx} u + u = 0$
- Eigenvalues: $\pm\sqrt{1 - n^2\omega^2}$, $n \geq 1$.
- The number of hyperbolic eigenvalues is always finite and increases when $\omega \to 0$.

Breather breakdown from spatial dynamics point of view

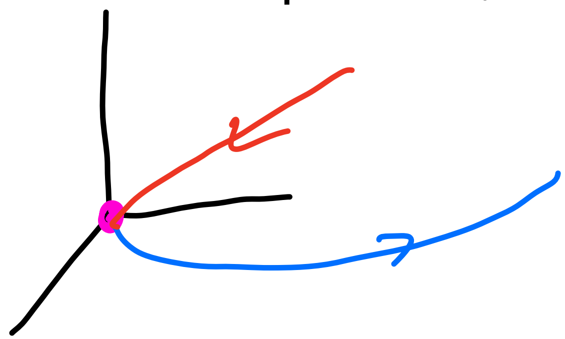
- Eigenvalues: $\pm\sqrt{1 - n^2\omega^2}$, $n \geq 1$.

- Bifurcations: At $\omega = \frac{1}{k}$, $k \in \mathbb{N}$, a new pair of (weakly) hyperbolic eigenvalues appears. small homoclinic orbits can appear!

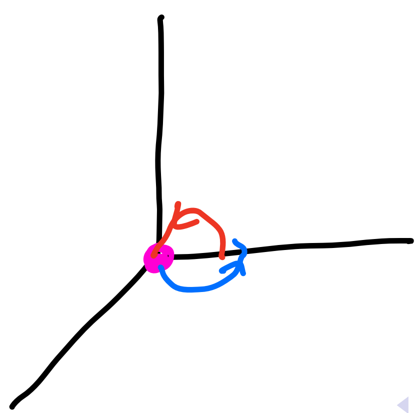
- Two settings:
  - Close to bifurcation: $0 < \frac{1}{k} - \omega \ll 1$, $k \in \mathbb{N}$.
  - Far from bifurcation: Otherwise.

Far from bifurcation:

- Far from bifurcation: All hyperbolic eigenvalues are "strong".
- All orbits in the stable/unstable invariant manifolds of $u = 0$ escape "far away" from $u = 0$.
- If homoclinic loops exist, they must be large.



- Small homoclinic loops may only appear when ω is close to bifurcation ($0 < \frac{1}{k} - \omega \ll 1$).



Close to the first bifurcation

- Kruskal-Segur setting: Close to the first bifurcation i.e.

$$0 < 1 - \omega \ll 1.$$

- Key setting: Close to the first bifurcation in the odd in t setting:

$$u(x,\tau) = \sum_{n\geq 1} u_n(x)\sin(n\tau).$$

- For the first bifurcation: take

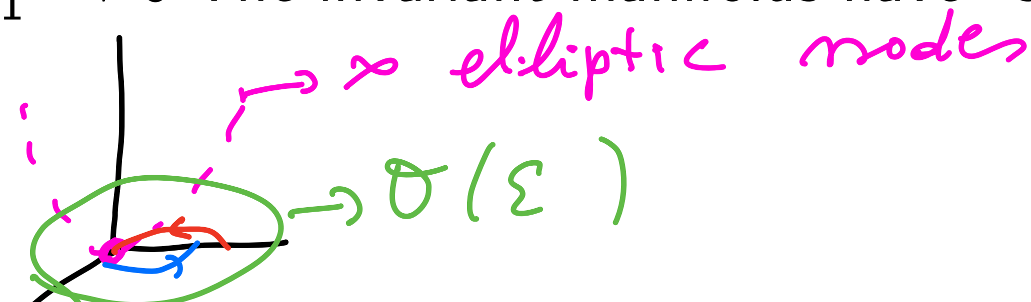
$$\omega = \sqrt{1-\varepsilon^2} \qquad \text{with} \qquad 0 < \varepsilon \ll 1.$$

- The other cases $\omega \simeq \frac{1}{k}$ can be proven using the results for this setting.

First bifurcation in the odd in τ setting

$$\omega^2 \partial_{\tau\tau} u - \partial_{xx} u + u - \frac{1}{3} u^3 - f(u) = 0, \quad f(u) = \mathcal{O}(u^5), \text{ odd analytic}$$

- $\omega^2 = 1 - \varepsilon^2$
- Eigenvalues: $\lambda_1^{\pm} = \pm\varepsilon$ and $\lambda_n^{\pm} = \pm i\sqrt{n^2(1-\varepsilon^2) - 1}$, $n \geq 2$.
- Spatial dynamics (x as evolution variable): one dimensional (weak) stable and unstable invariant manifolds of $u = 0$.
- Weakness $\lambda_1^{\pm} \to 0$ The invariant manifolds have "size" $\mathcal{O}(\varepsilon)$.



- Scaling: $u = \varepsilon v$, and "time" $y = \varepsilon x$

$$\partial_y^2 v - \frac{\omega^2}{\varepsilon^2} \partial_\tau^2 v - \frac{1}{\varepsilon^2} v + \frac{1}{3} v^3 + \frac{1}{\varepsilon^3} f(\varepsilon v) = 0,$$

Equation for the Fourier coefficients (odd setting)

Writing the equations of the Fourier coefficients: $v(y,\tau) = \sum v_n(y) \sin n\tau$, we obtain a singularly perturbed problem:

$$\begin{cases} \ddot{v}_1 = v_1 - \Pi_1 \left[\dfrac{v^3}{3} + \mathcal{O}(\varepsilon^2) \right], \\ \varepsilon^2 \ddot{v}_n = -\mu_n^2 v_n - \varepsilon^2 \Pi_n \left[\dfrac{v^3}{3} + \mathcal{O}(\varepsilon^2) \right], \ n \geq 2, \end{cases}$$

with $\cdot = d/dy$ and $\mu_n = \sqrt{n^2(1-\varepsilon^2) - 1}$.

$v = 0$ is a saddle center point with infinitely many elliptic directions.

- Taking the singular limit $\varepsilon \to 0$, the critical manifold is the plane

$$\mathcal{M} = \{v_n = 0, \ \dot{v}_n = 0, \ n > 1\}.$$

which is normally elliptic: normal eigenvalues $\pm i \frac{\mu_n}{\varepsilon^2} \to$ fast oscillations!

A normally elliptic slow manifold

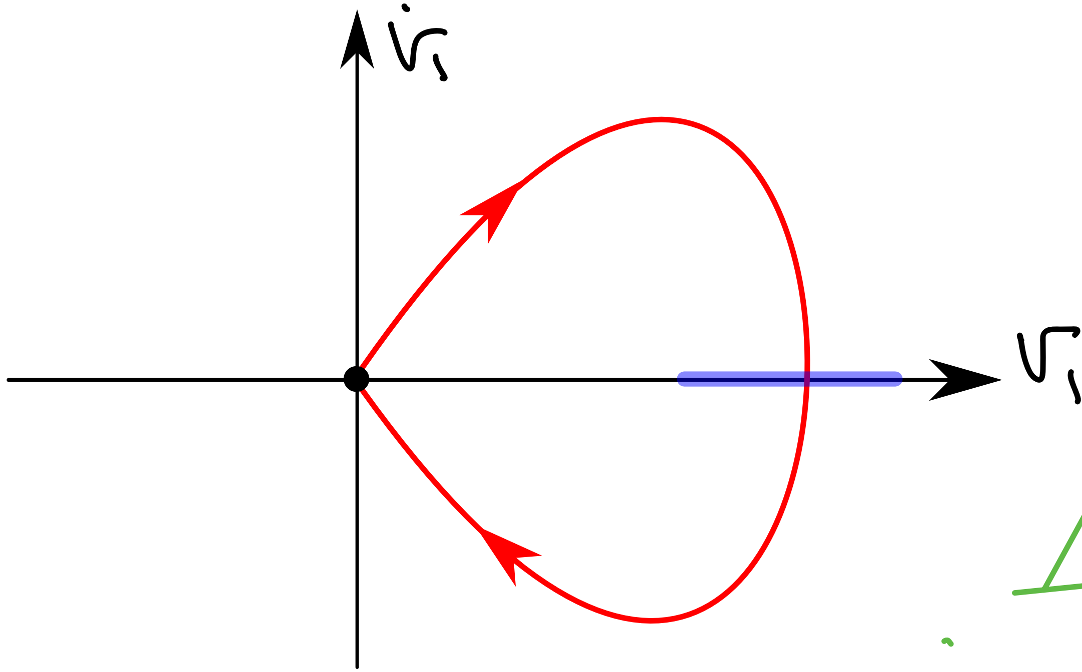
$\mathcal{M} = \{v_n = 0,\ \dot{v}_n = 0,\ n > 1\}.$

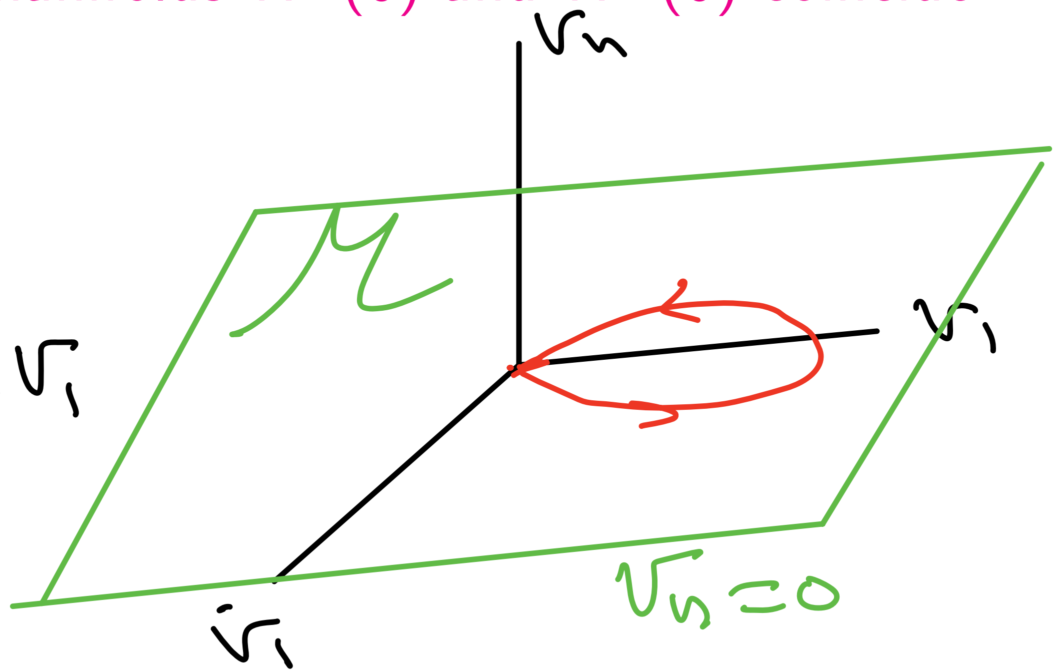
The dynamics in $\mathcal{M}$ is given by the Duffing equation: $\ddot{v}_1 = v_1 - \dfrac{v_1^3}{4}$.

- Limit equation has a homoclinic orbit to $v_1 = \dot{v}_1 = 0$.

$$v_1^h(y,\tau) = \frac{2\sqrt{2}}{\cosh(y)}, \quad v_n^h = 0,\ n \geq 2.$$

- In the limit problem, the invariant manifolds $W^s(0)$ and $W^u(0)$ coincide.





$\varepsilon \neq 0$: Homoclinic breakdown

- Does the homoclinic orbit persist for the full problem?
- It is a singular perturbation problem:

$$\begin{array}{r}\text{Fast rotation versus} \\ \text{weak hyperbolicity}\end{array} \longrightarrow \text{Exponentially small phenomena.}$$

- Hard to measure the distance between the one dimensional perturbed invariant manifolds $W^s(0)$ and $W^u(0)$.
- Classical perturbative methods (Melnikov Theory) cannot be applied.

$\varepsilon > 0$: Formal series expansions

- Look for parameterizations of $W^u(0)$ and $W^s(0)$

$$v_n^u(y,\varepsilon),\ \ v_n^s(y,\varepsilon),\ \ n \geq 1$$

satisfying:

$$\lim_{y\to-\infty} v_n^u(y,\varepsilon) = 0, \qquad \lim_{y\to+\infty} v_n^s(y,\varepsilon) = 0$$

- Look for formal solutions as formal power series of ε:

$$v_n^*(y,\varepsilon) = v_{n,0}(y) + \varepsilon v_{n,1}^*(y) + \varepsilon^2 v_{n,2}^*(y) + \dots \text{ for } * = s, u$$

- One can check:

$$v_{n,k}^u(y) = v_{n,k}^s(y)\ \ \forall k \in \mathbb{N}$$

- Thus: their difference is beyond all orders:

$$v_n^u(y,\varepsilon) - v_n^s(y,\varepsilon) = \mathcal{O}(\varepsilon^m)\ \ \forall m \in \mathbb{N}.$$

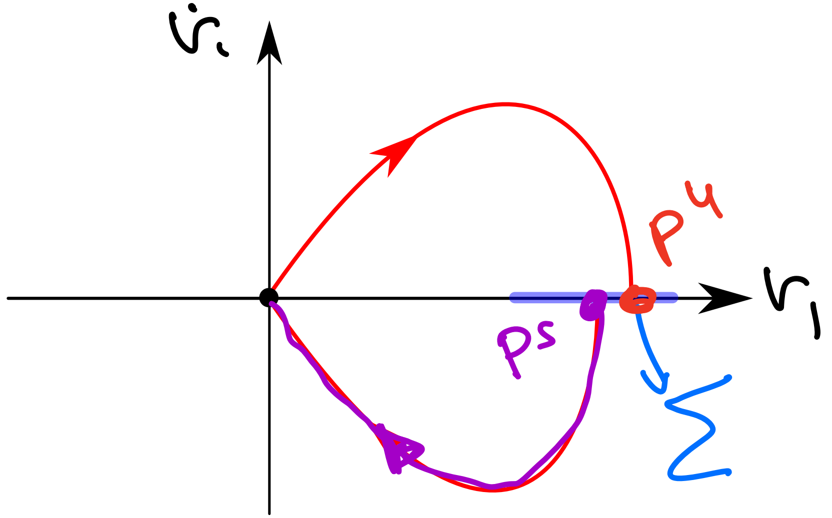
Kruskal and Segur work

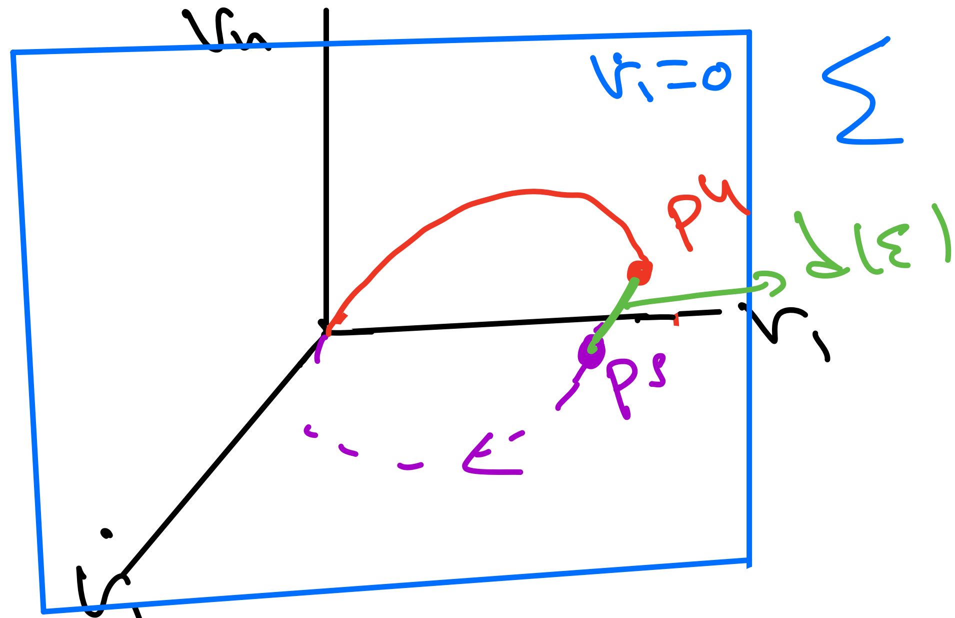
We have two possibilities:

1. The power series in ε are convergent and the manifolds coincide. In this case breathers exist!
2. The power series in ε are divergent and the difference between manifolds is flat with respect to ε.

- Typically, we expect the second case to happen.
- H. Segur, M.D. Kruskal. (1987) gave formal arguments which indicate that the series is not convergent and that there is breakdown.
- Question:

  How to make rigorous the formal arguments to show breathers breakdown.





Take a section transversal to the solutions

$$\Sigma = \{(v, \partial_y v);\ \mathcal{H}(v, \partial_y v) = 0 \quad \text{and} \quad \Pi_1 [\partial_y v] = 0\}$$

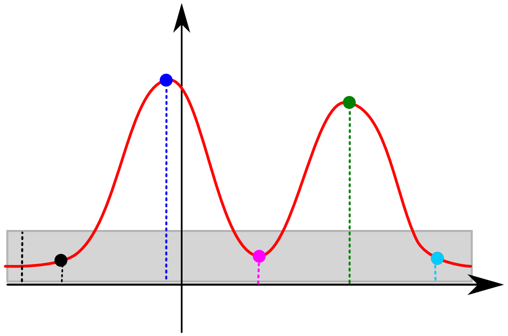
Theorem (G.-Gomide-Seara-Zeng)

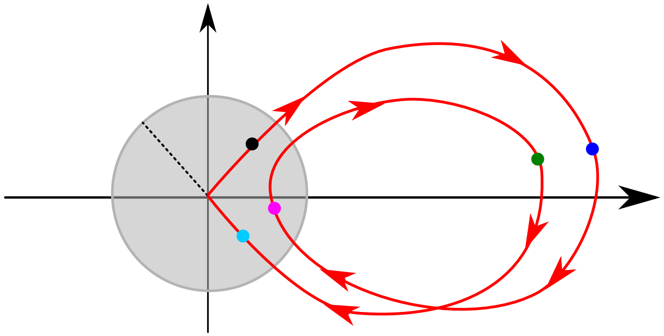
$P^{uns,st}$ the first intersection points of $W^{uns,st}$ with Σ. There exists a constant Θ_f such that, for $\varepsilon \ll 1$, the distance $d(\varepsilon) = P^{uns} - P^{st}$ satisfies

$$\Pi_3[d(\varepsilon)] = \frac{2}{\varepsilon} e^{-\frac{\pi\sqrt{2}}{\varepsilon}} \left(\Theta_f + \mathcal{O}\left(1/\log\varepsilon\right)\right) \quad \Pi_n[d(\varepsilon)] = \frac{2}{\varepsilon} e^{-\frac{\pi\sqrt{2}}{\varepsilon}} \mathcal{O}\left(1/\log\varepsilon\right) \ \ n > 3.$$

Implication on breathers (if $\Theta_f \neq 0$)

- The constant Θ_f is the one appearing in the main theorems.
- If $\Theta_f \neq 0$, then the invariant manifolds $W^-(0)$ and $W^+(0)$ do not intersect the first time they reach Σ.
- It rules out the existence of homoclinics continuation of those of the singular limit problem (single-bump homoclinic loops).
- Even if $\Theta_f \neq 0$, $W^-(0)$, $W^+(0)$ may still coincide after more rounds. This would give multi-bump breathers.





This analysis is the starting point to deal with all bifurcations.

Some ideas about the proof of the first bifurcation theorem

- Exponentially small splitting of separatrices.
- We follow the ideas by Lazutkin for the homoclinic breakdown for the Standard Map (also Kruskal and Segur).
- Mostly been applied to:
  - 2- dimensional area preserving maps
  - Invariant manifolds of periodic orbits or invariant tori at resonances of nearly integrable Hamiltonian systems (Arnold diffusion)
  - Local bifurcations for Hamiltonian/Reversible/Volume preserving systems

Analytic continuation to complex domains

- Homoclinic for the singular limit: $v_h(y,\tau) = \dfrac{2\sqrt{2}}{\cosh(y)} \sin\tau$.
- Look for stable/unstable solutions v^{uns} and v^{st} of Klein-Gordon eq.
- $v^{\mathrm{uns}}, v^{\mathrm{st}}$ are exponentially close to each other.
- It is very difficult to study the difference.
- v_h has singularities at $y = \pm i\pi/2$, therefore it blows up.
- Extend $v^{\mathrm{uns}}, v^{\mathrm{st}}$ to complex y up to $y \sim \pm i\pi/2$.
- v_h blows up at $y = \pm i\pi/2 \to v^{\mathrm{uns}}, v^{\mathrm{st}}$ should be large for $y \sim \pm i\pi/2$
- Its difference is easier to measure at $y \sim \pm i\pi/2$.

Analytic continuation to complex domains

- Lazutkin and Kruskal & Segur: Analyze the difference between v^{uns} and v^{st} when $y \mp i\pi/2 \sim \varepsilon$.
- When $y \mp i\pi/2 \sim \varepsilon$, $v^{\text{uns}}, v^{\text{st}}$ are not well approximated by the unperturbed homoclinic v_h.
- Singular change: $z = \varepsilon^{-1}\left(y - i\frac{\pi}{2}\right)$ and $\phi(z,\tau) = \varepsilon v\left(i\frac{\pi}{2} + \varepsilon z, \tau\right)$.
- Let $\varepsilon \to 0$ and we get a new equation for the first order: the inner equation

$$\partial_z^2\phi - \partial_\tau^2\phi - \phi + \frac{1}{3}\phi^3 + f(\phi) = 0$$

Analytic continuation to complex domains

- The analysis of suitable solutions of the inner equation and their difference provides the constant Θ_f appearing in the distance formula.

- Θ_f is a Stokes constant (Borel Resummation, Resurgence Theory).

- Θ_f depends on the full jet of the nonlinearity f.
- Only one condition $\Theta_f \neq 0$ rules out breathers of any frequency!

Thank you for your attention