Com serien els Sistemes Dinàmics si no coneguéssim el Teorema de Sharkovskii

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Objectius

- L'enunciat del Teorema de Sharkovskiĭ és bonic, elegant i curiós. Però serveix d'alguna cosa?
- Certament les conseqüències directes del seu enunciat valen per situacions restringides.
- Però introdueix la noció d'estructura periòdica útil en situacions insospitades.
- Pel que fa a la tecnologia (moderna) de la seva demostració, introdueix nocions fonamentals:
  - Patterns que són l'esquelet de la dinàmica combinatòria en dimensió 1 i en homeos de superfícies (Thurston, Bestvina-Handel, ...)
  - Dinàmica combinatòria que caracteritza completament la dinàmica a tots nivells: Estructura periòdica, dinàmica topologica (transitivitat), dinàmica estadística (entropía topològica), ...

El Teorema de Sharkovskiĭ: Un enunciat sorprenent

We start by introducing

The Sharkovskii Ordering ${}_{\text{Sh}}\geq$:

$3 \;{}_{\text{Sh}}\!> 5 \;{}_{\text{Sh}}\!> 7 \;{}_{\text{Sh}}\!> \cdots \;{}_{\text{Sh}}\!> 2\cdot 3 \;{}_{\text{Sh}}\!> 2\cdot 5 \;{}_{\text{Sh}}\!> 2\cdot 7 \;{}_{\text{Sh}}\!> \cdots \;{}_{\text{Sh}}\!>$
$4\cdot 3 \;{}_{\text{Sh}}\!> 4\cdot 5 \;{}_{\text{Sh}}\!> 4\cdot 7 \;{}_{\text{Sh}}\!> \cdots \;{}_{\text{Sh}}\!> \cdots \;{}_{\text{Sh}}\!>$
$2^n\cdot 3 \;{}_{\text{Sh}}\!> 2^n\cdot 5 \;{}_{\text{Sh}}\!> 2^n\cdot 7 \;{}_{\text{Sh}}\!> \cdots \;{}_{\text{Sh}}\!> 2^\infty \;{}_{\text{Sh}}\!> \cdots \;{}_{\text{Sh}}\!>$
$2^n \;{}_{\text{Sh}}\!> \cdots \;{}_{\text{Sh}}\!> 16 \;{}_{\text{Sh}}\!> 8 \;{}_{\text{Sh}}\!> 4 \;{}_{\text{Sh}}\!> 2 \;{}_{\text{Sh}}\!> 1.$

is defined on the set $\mathbb{N}_{\text{Sh}} = \mathbb{N} \cup \{2^\infty\}$
(we have to include the symbol 2^∞ to assure the existence of supremum for certain sets).

In the ordering ${}_{\text{Sh}}\!>$ the least element is 1 and the largest is 3.

The supremum of the set $\{1, 2, 4, \dots, 2^n, \dots\}$ is 2^∞.

The Sharkovskiĭ Ordering formal definition

If $k = k' \cdot 2^p$ where p is non negative and k' is odd:

1. $k \;_{\text{Sh}}\!> 2^\infty$ if $k' > 1$,
2. $2^\infty \;_{\text{Sh}}\!> k$ if $k' = 1$,

and if $n = n' \cdot 2^q$ where q is non negative and n' is odd, then $n \;_{\text{Sh}}\!> k$ if and only if one of the following next statements holds:

3. $k' > 1$, $n' > 1$ and $p > q$,
4. $k' > n' > 1$ and $p = q$,
5. $k' = 1$ and $n' > 1$,
6. $k' = 1$, $n' = 1$ and $p < q$.

For $s \in \mathbb{N}_{\text{Sh}}$, $\text{S}_{\text{sh}}(s)$ denotes the set $\{k \in \mathbb{N} : s \;_{\text{Sh}}\!\geq k\}$.

Examples of sets of the form $\text{S}_{\text{sh}}(s)$ are:

- $\text{S}_{\text{sh}}(2^\infty) = \{1, 2, 4, \ldots, 2^n, \ldots\}$,
- $\text{S}_{\text{sh}}(3) = \mathbb{N}$,
- $\text{S}_{\text{sh}}(6)$ is the set of all positive even numbers union $\{1\}$, and
- $\text{S}_{\text{sh}}(16) = \{1, 2, 4, 8, 16\}$.

Remark

$\text{S}_{\text{sh}}(s)$ is finite if and only if $s \in \text{S}_{\text{sh}}(2^\infty)$.

Theorem (Sharkovskii)

For each continuous map g from a closed interval of the real line into itself, there exists $s \in \mathbb{N}_{\mathsf{Sh}}$ such that $\operatorname{Per}(g) = \mathsf{S}_{\mathsf{sh}}(s)$.
Conversely, for each $s \in \mathbb{N}_{\mathsf{Sh}}$ there exists a continuous map g_s from a closed interval of the real line into itself such that $\operatorname{Per}(g_s) = \mathsf{S}_{\mathsf{sh}}(s)$.

$\operatorname{Per}(g)$ denotes the set of (least) periods of all periodic points of g.

A consequence of Sharkovskii Theorem Statement: Triangular maps

A triangular map on an n-dimensional rectangle $\mathcal{Q}$ is defined as

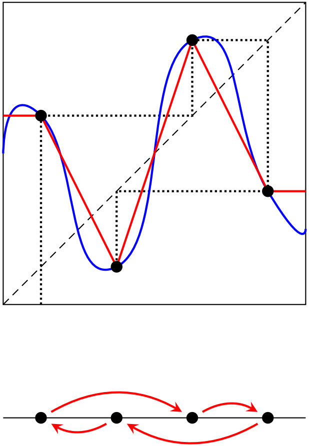
$$\begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{pmatrix} \longmapsto \begin{pmatrix} f_1(x_1) \\ f_2(x_1, x_2) \\ \vdots \\ f_n(x_1, x_2, \ldots, x_n) \end{pmatrix}$$

Theorem (Kolyada)

Sharkovskii Theorem holds for triangular maps on an n-dimensional rectangles.

Statement: The periodic structure

- If a map has period three it has periodic points of all periods (*Period three implies chaos*).
- If a map has no period 2 (and this is easy to check) then only has fixed points.
- More general example: If 3-star map has no periods 2 and 3, then only has fixed points.
- And many more . . . . . .



The red map is called the *connect the dots map*. One has

Per(of blue map) $\supset$ Per(red map).

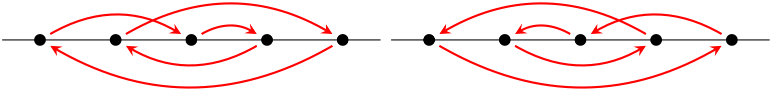
A crucial idea: the permutation aka *pattern*

Let us suppose, *for example*, that P is an orbit of *Stefan type* of period n. That is, of the following type:

$$p_n < p_{n-2} < \cdots < p_5 < p_3 < p_1 < p_2 < p_4 < \cdots < p_{n-3} < p_{n-1},$$

or

$$p_{n-1} < p_{n-3} < \cdots < p_4 < p_2 < p_1 < p_3 < p_5 < \cdots < p_{n-2} < p_n.$$



Markov Graph

Lemma

The vertices of the f_P-(combinatorial) Markov graph of f associated to P can be labelled so that their arrows are

1. $I_1 \longrightarrow I_2 \longrightarrow \cdots \longrightarrow I_{s-1} \longrightarrow I_1,$
2. $I_1 \longrightarrow I_1,$
3. $I_{s-1} \longrightarrow I_1, \;\; I_{s-1} \longrightarrow I_3, \;\; I_{s-1} \longrightarrow I_5, \;\; \dots, \;\; I_{s-1} \longrightarrow I_{s-2}.$

Conclusion

It is easy to see that the previous Markov Graph gives loops of length equals to any positive integer contained in $S(n)$. Consequently, $S(n) \subset \mathrm{Per}(f_P)$, since:

Lemma

Let $f \in \mathcal{C}^0(I, I)$, let $P \subset I$ be a finite set and let $\alpha = I_1 \longrightarrow I_2 \longrightarrow \cdots \longrightarrow I_{s-1} \longrightarrow I_1$ be a loop in the f-Markov graph associated to P. Then, there exists a fixed point x of f^n, such that $f^i(x) \in I_i$ for $i = 0, 1, \dots, n-1$. By choosing the loop in an appropriate way one can show that I_1 contains a point x whose (least) period is precisely n. Consequently, $n \in \mathrm{Per}(f)$.

A philosophical consequence of the proof of Sharkovskii Theorem: Symbolic Dynamics; a rich world

- The symbolic dynamics associated to the markov graph of a connect the dots map characterises completely the dynamics of the maps.
- In particular it allows to study transitivity and other topological dynamics properties.
- Thus gives lower bounds of the number of fixed points.
- Thus gives lower bounds of the topological entropy (Perron-Frobenius Theorem and Power Method to compute spectral radius play an important theoreticalrole here).
- Thus gives lower bounds of $\cdots$ (in what are you interested in)

Patterns and their consequences

1. f_π minimises topological entropy within the class of interval maps admitting a periodic orbit whose pattern is π.
2. f_π admits a Markov partition which gives a good "coding" to describe the dynamics of the map f_π. The topological entropy of f_π may be calculated from this partition.
3. f_π is essentially unique.
4. the pattern of A forces a pattern ρ if and only if f_π has a periodic orbit whose pattern is ρ. We recall that a pattern A *forces* a pattern B if and only if each map exhibiting the pattern A also exhibits the pattern B. In this sense, the dynamics of f_π are minimal within the class of maps admitting a periodic orbit whose pattern is π_A.

Patterns and their consequences

A summary of three known cases

PERIODIC ORBIT OF	PATTERN A	CANONICAL REPRESENTATIVES
interval map	permutation π induced by map on orbit	'Connect-the-dots' maps f_π
tree map	'relative positions' of the points of orbit	canonical models of trees
surface homeo.	braid type (isotopy class rel. orbit)	Nielsen-Thurston representatives