FUNCIONES ENTERAS CON DOMINIOS ERRANTES

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FATOU AND JULIA SETS

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- **Julia set**: locus of chaotic behaviour. $J(f) = \mathbb{C} \setminus F(f)$.

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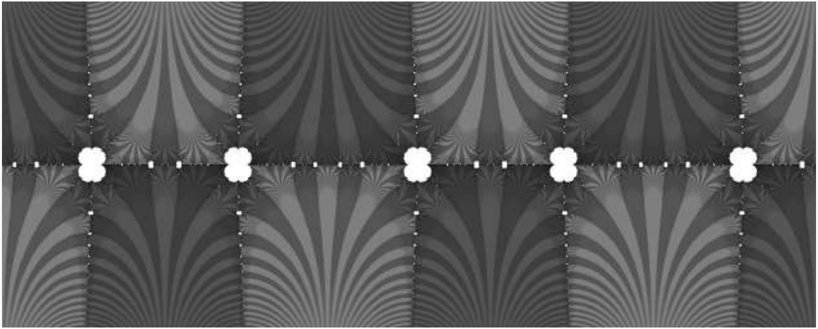
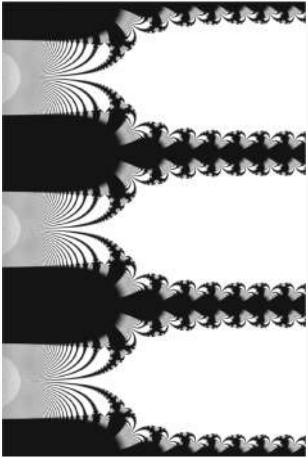
  - Baker (1976) provided the first example.
  - Many others have been found since then.



$$f(z) = z + \sin(z) + 2\pi.$$

*Figure by L. Rempe. Wikipedia commons



$$f(z) = z - 1 + e^{-z} + 2\pi i.$$

*Picture from [Rempe-Sixsmith, 17']

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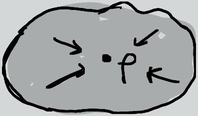
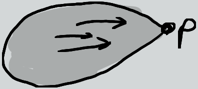


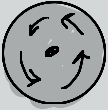
attracting basin

$f^k(z) \to p \in U, |f'(p)| < 1.$



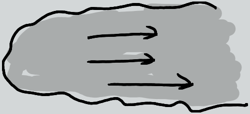
parabolic basin

$f^k(z) \to p \in \partial U, f'(p) = 1.$



Siegel disc

$f|_U$ conjugate to $e^{i\vartheta}z$, $\vartheta \in \mathbb{R} \setminus \mathbb{Q}$.



Baker domain

$f^k(z) \to \infty$, ∞ essential sing.

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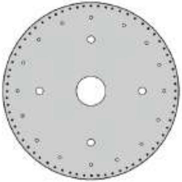


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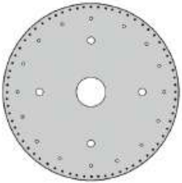


- Where do they go?
  - Can they stay away from infinity?

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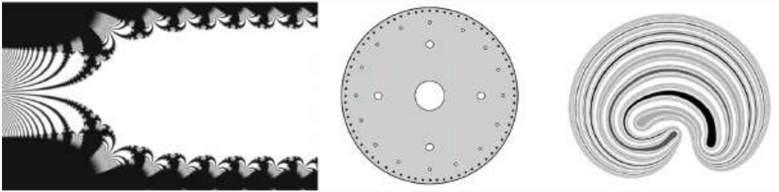

- Where do they go?
  - Can they stay away from infinity?
- For which functions do they occur?
  - Relation with singular values.

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WHERE DO THEY GO?

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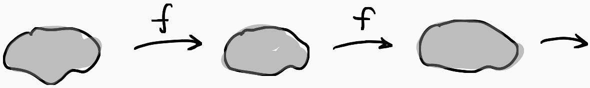
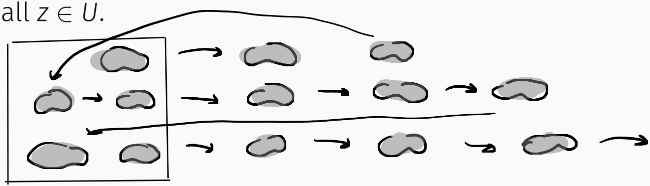
- **Escaping**: $f^n(z) \to \infty$ as $n \to \infty$ for all $z \in U$.



- **Oscillating**: there are subsequences $f^{n_k}(z) \to \infty$ and $f^{m_k}(z) \not\to \infty$ for all $z \in U$.



- **Bounded-orbit**: $f^{n_k}(z) \not\to \infty$ for all $z \in U$ and all subseq. (n_k).
  - Do they exist?
  - Stronger version: is there U such that $f^n(U) \subset D$ for all n and some bounded domain D?

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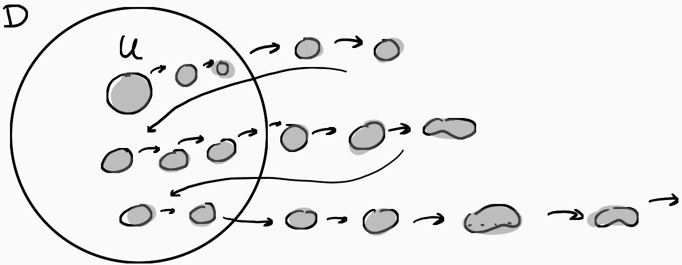
  - **Lakes of Wada** example. [MartíPete-Rempe-Waterman '22].

FIRST RESULT: NEARLY BOUNDED ORBIT

Theorem A (P.-Sixsmith, ’23)

There exists an entire function f with a wandering domain U and a bounded domain D such that

$$\lim_{k\to\infty} \frac{\#\{n \leq k : f^n(U) \subset D\}}{k} = 1.$$



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- Let $(n_j)_{j\in\mathbb{N}}, (m_j)_{j\in\mathbb{N}}$ be sequences of natural numbers. We can *prescribe* the iterates U spends on D:

  For each $n \in \mathbb{N}$, $f^n(U) \subset D$ if and only if

$$\sum_{i=0}^{p}(n_i + m_i) < n \le \sum_{i=0}^{p}(n_i + m_i) + n_{p+1} \quad \text{for some } p \ge 0.$$

Runge's theorem

Let A be a full compact set. Let $h\colon A \to \mathbb{C}$ be a holomorphic function. Then for every $\varepsilon > 0$, there exists a polynomial f such that

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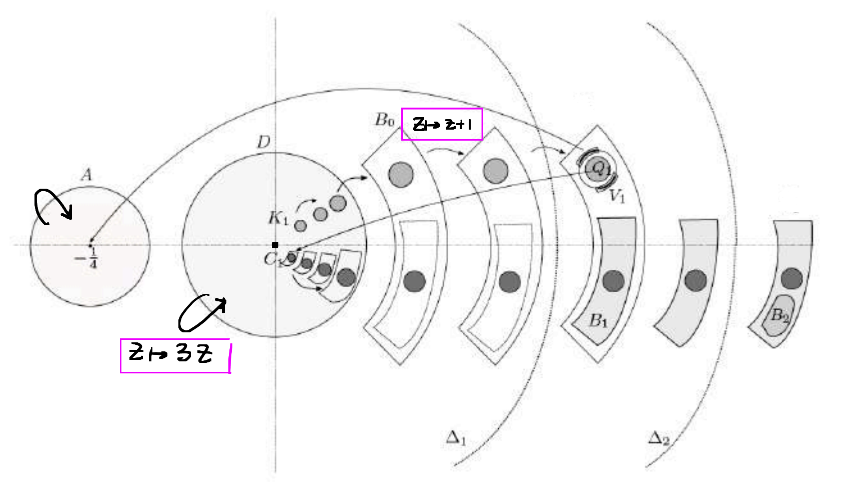
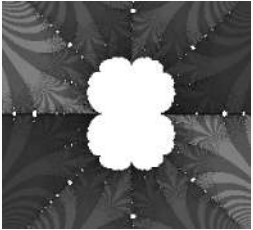
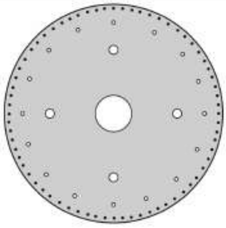
  - $\min\{|z|\colon z \in A_n\} \to \infty$ as $n \to \infty$.
- Finitely many *conditions* can be prescribed for each k, that is, $f(z_{i,k}) = h_k(z_{i,k})$ and $f'(z_{i,k}) = h_k'(z_{i,k})$ for finitely many i.



We iteratively use Runge's theorem on a sequence (f_n) such that f_{n+1} approximates f_n on Δ_n.

	simply connected	multiply connected
bounded		
unbounded		Not possible. [Baker, 84]

FAST ESCAPING POINTS

Points may converge to infinity at different rates.

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Theorem (Rippon and Stallard, ’11)

Let f be a transcendental entire function. Given any $a_n \to \infty$, there exists $z \in I(f) \cap J(f)$ and $N \in \mathbb{N}$ such that $|f^n(z)| \leq a_n$ for all $n \geq N$.

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For a transcendental entire f, we define its **fast escaping set** as

$$A(f) := \{z \in \mathbb{C} : \text{there is } \ell \in \mathbb{N} \text{ such that } |f^{n+\ell}(z)| \geq M^n(R), \ \text{for } n \in \mathbb{N}\},$$

where $M(r) := \max_{|z|=r} |f(z)|$ and $R > 0$ is large enough.

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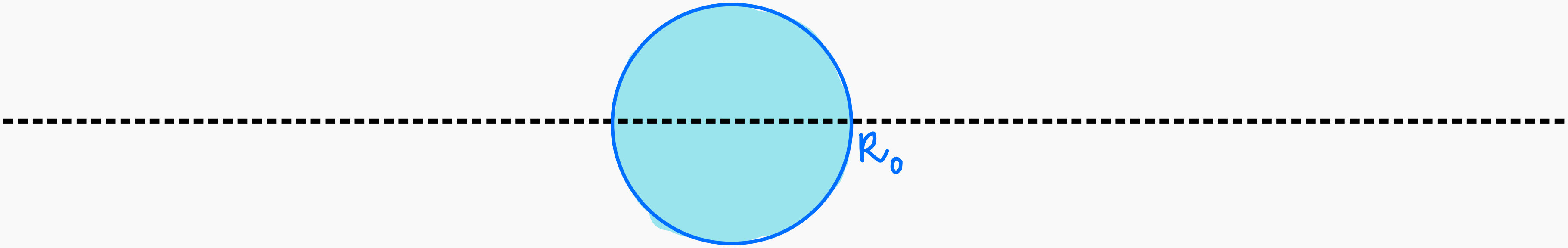

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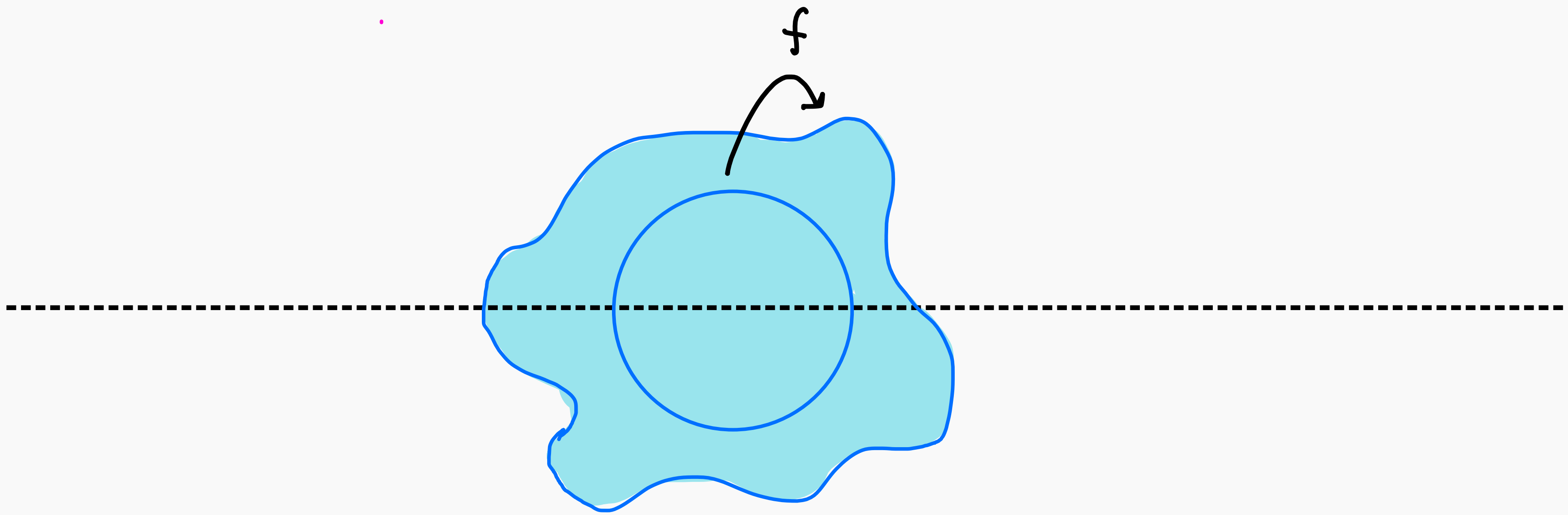

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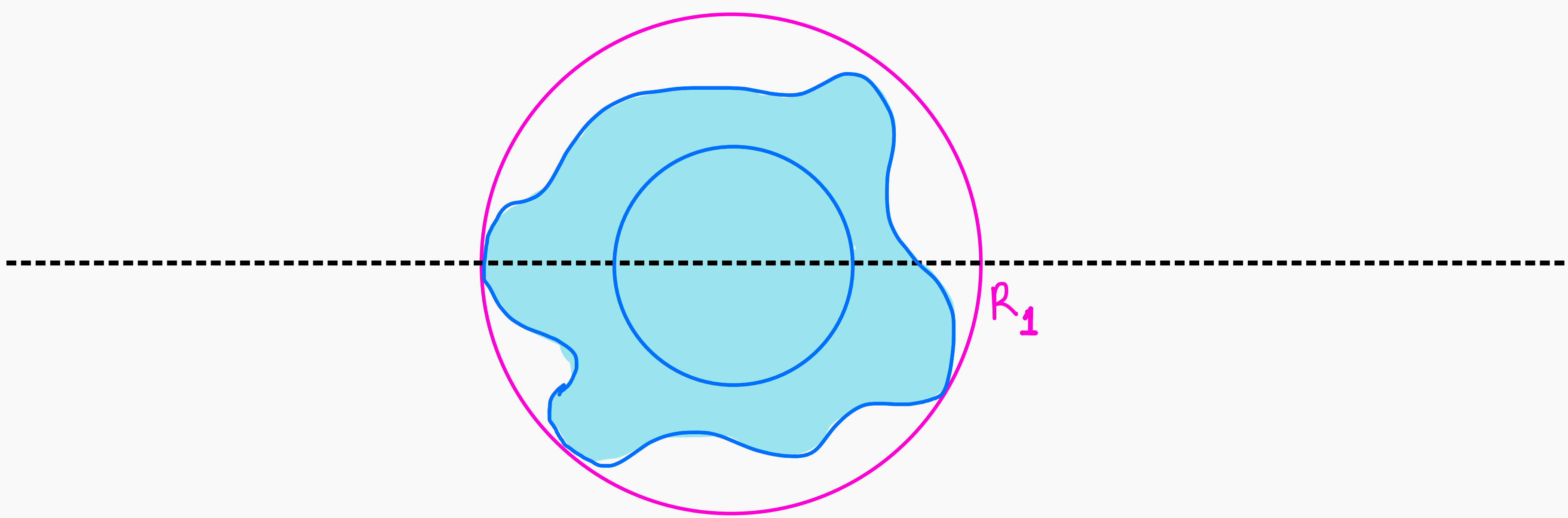

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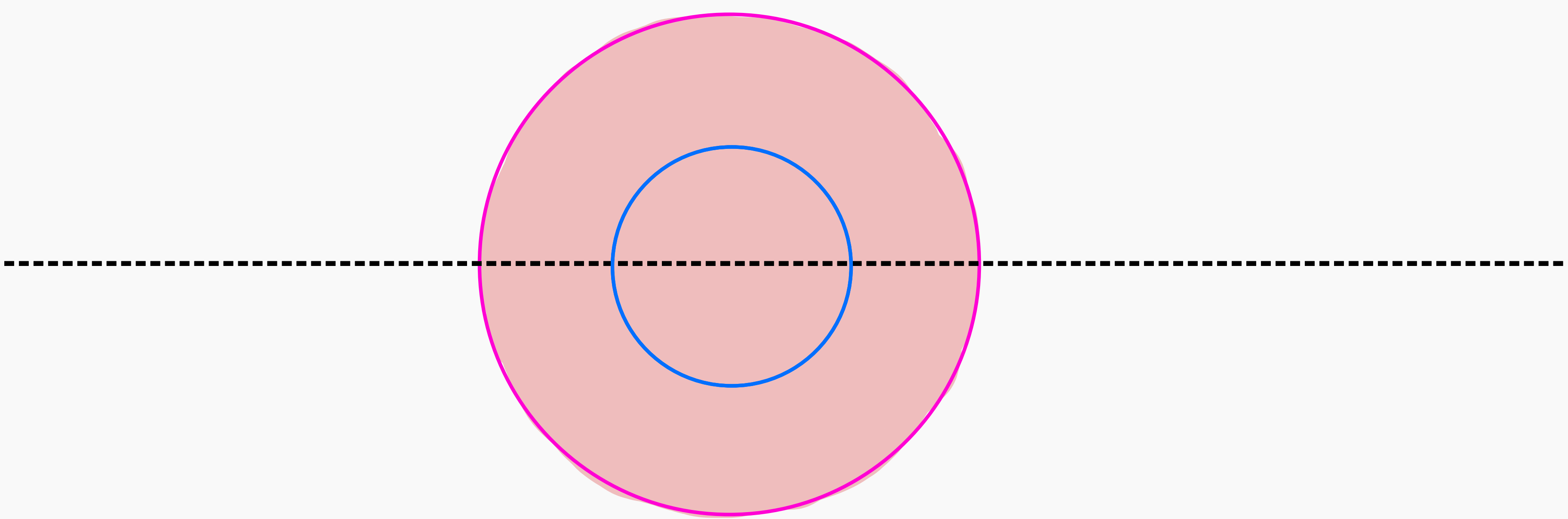

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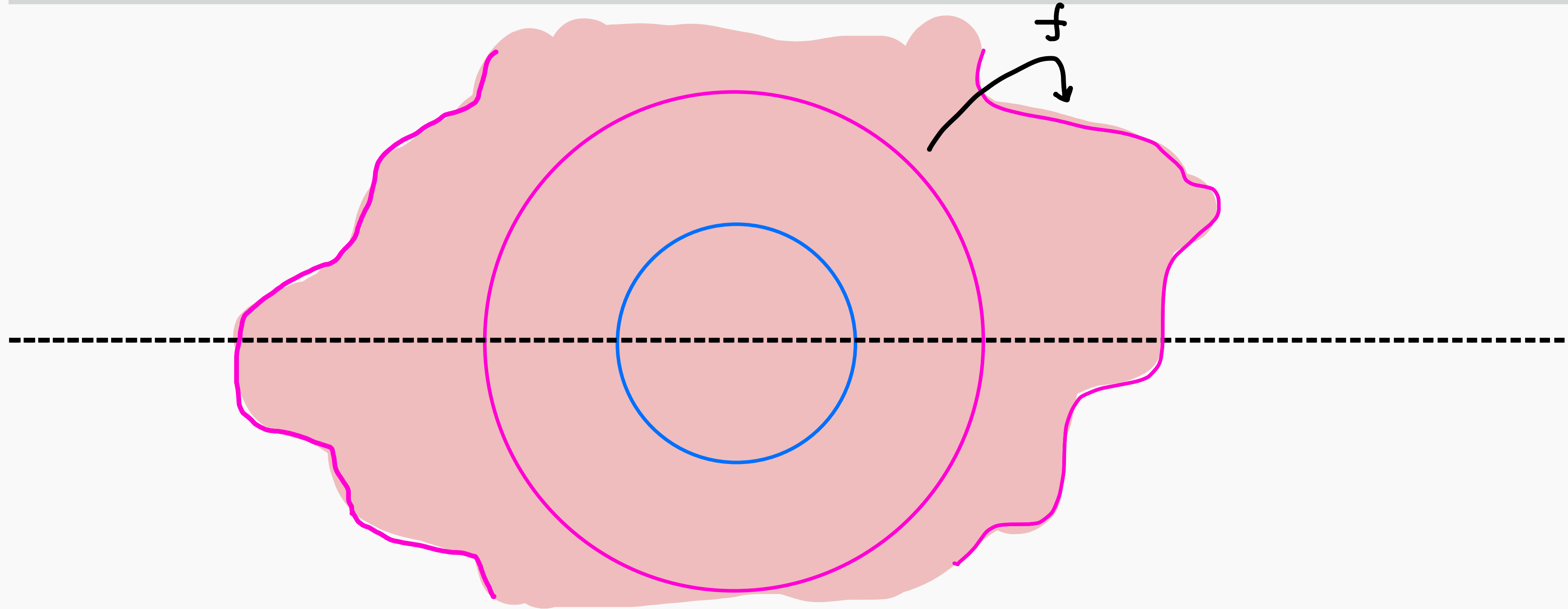

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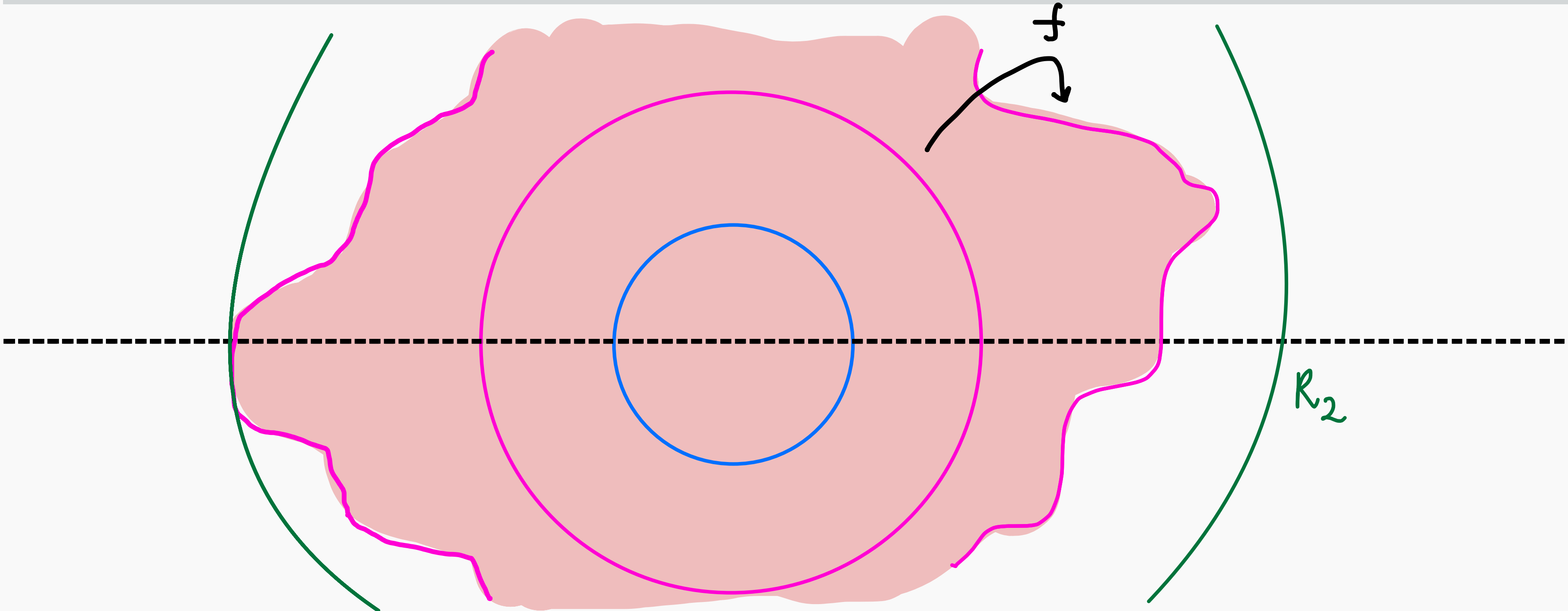

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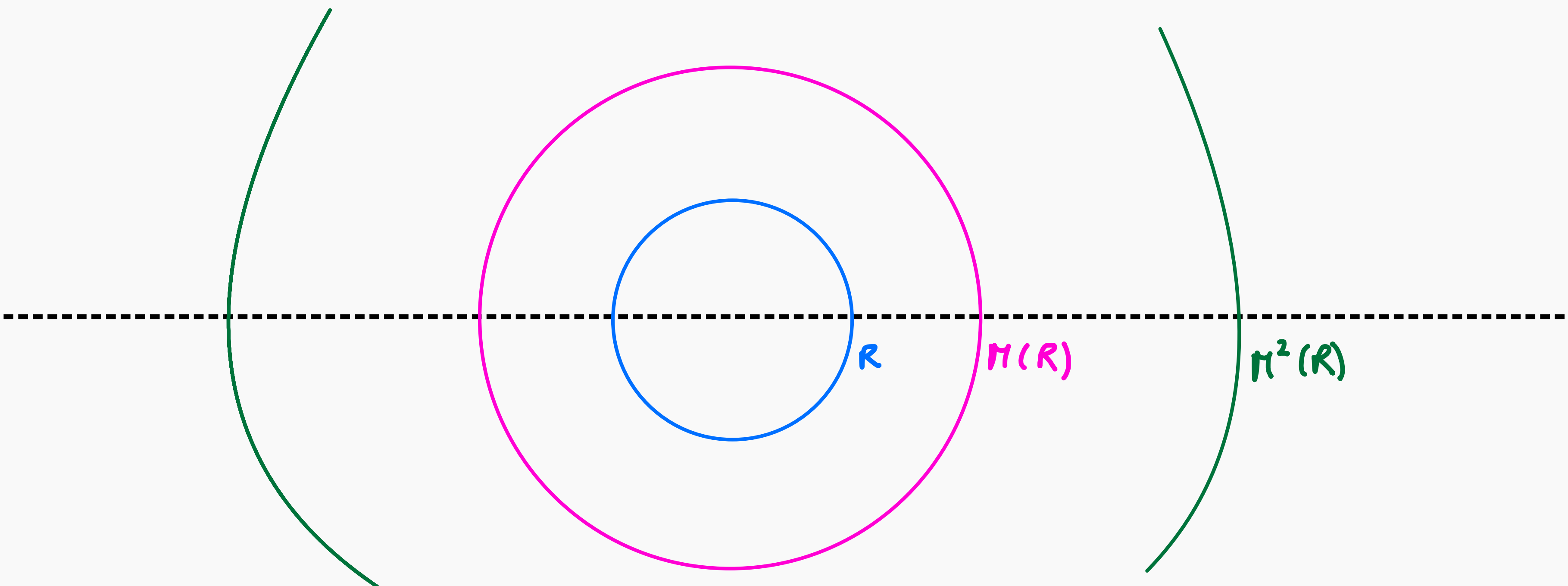

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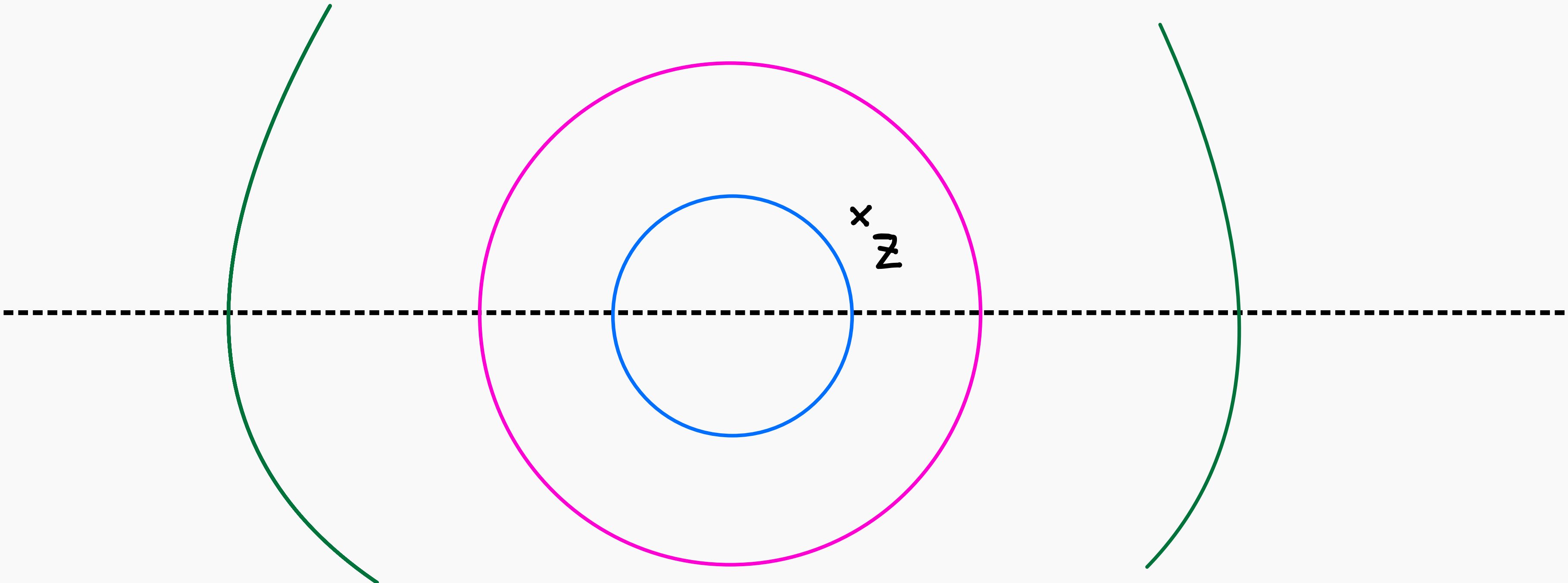
where $M_f(r) := \max_{|z|=r} |f(z)|$ and $R > 0$ is large enough.



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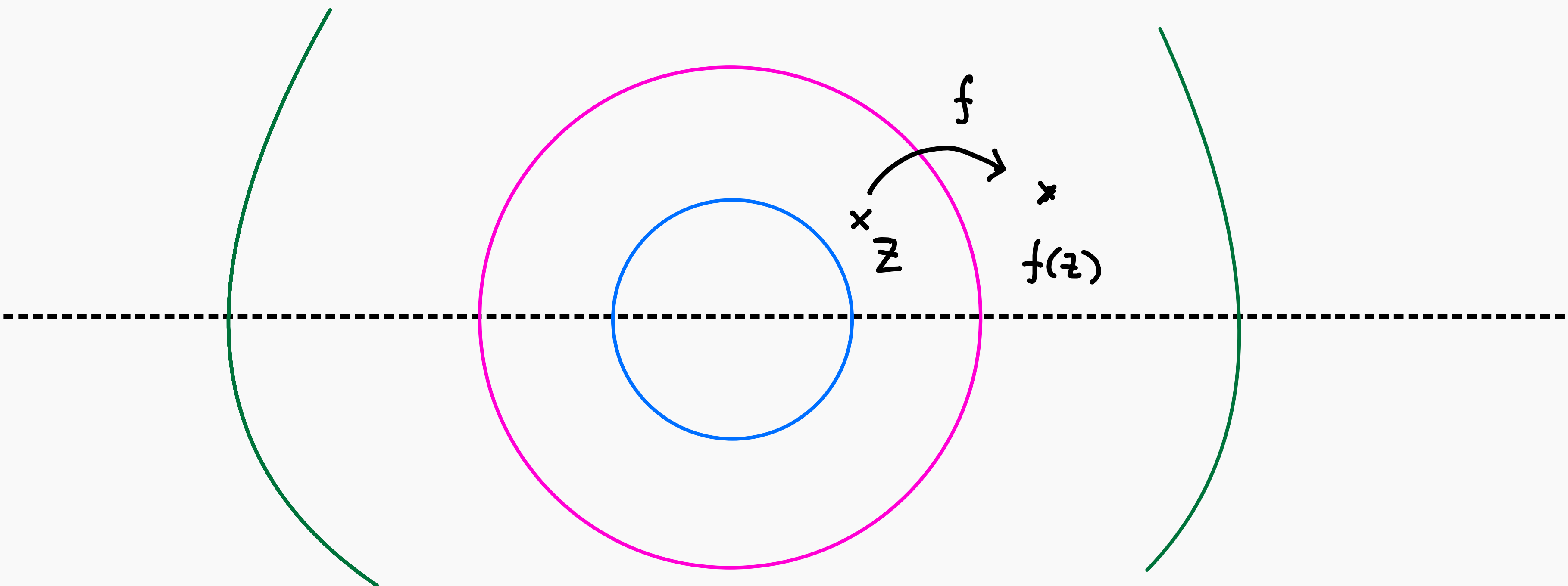

FAST ESCAPING POINTS

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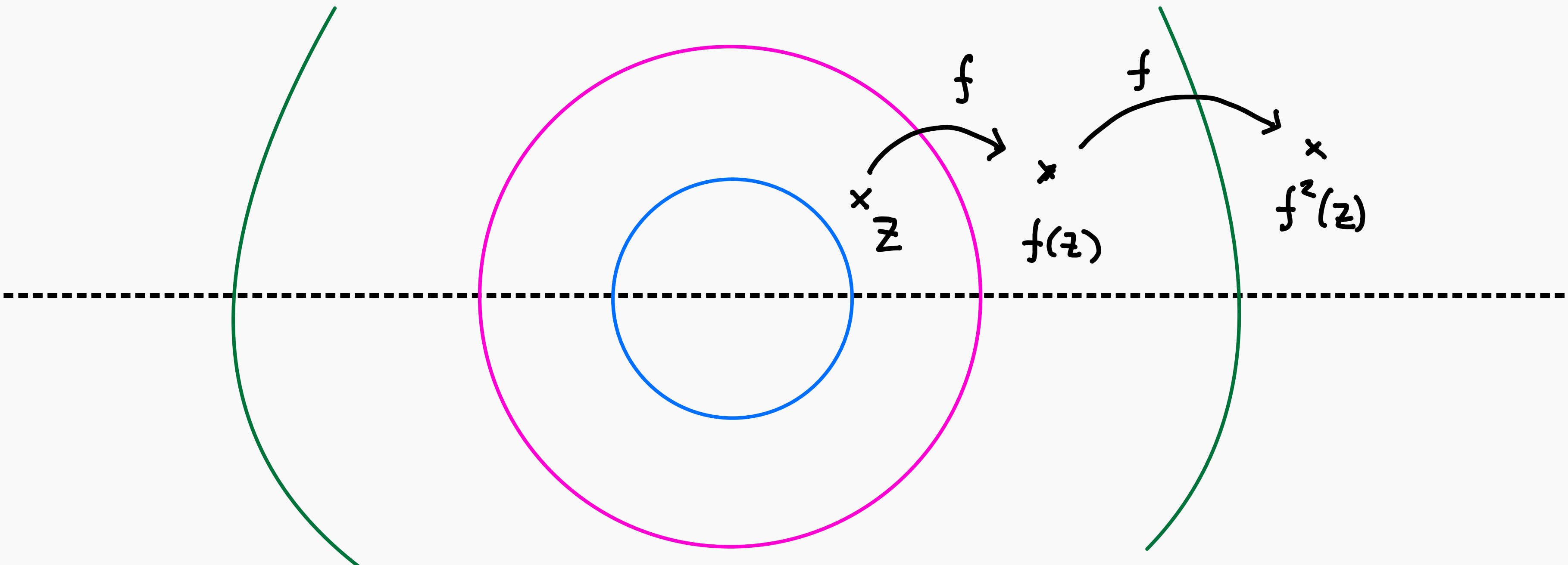

(fast) escaping fatou components

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⋆ Equivalently: Is there a t.e.f. with an unbounded fast escaping wandering domain?

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Theorem (Benini, Rippon and Stallard, '16)

If $f \circ g = g \circ f$ and f, g have no **simply connected** fast escaping wandering domains, then $J(f) = J(g)$.

Theorem B (Evdoridou-Glücksam-P., ’23)

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- Our function can be chosen to have different types of internal dynamics with respect to the boundary.
- Our method provides us with control on the growth, and so on the **order**, of the function.

$$\rho(f) := \limsup_{r\to\infty} \frac{\log\log M_f(r)}{\log r}.$$

Baker's conjecture, 1981

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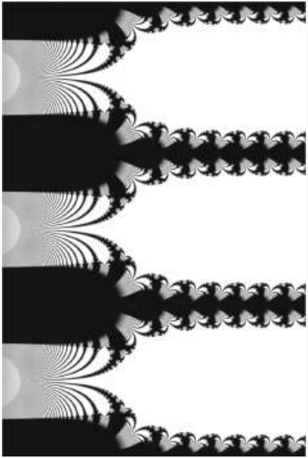

$$f(z) = z - 1 + e^{-z} + 2\pi i.$$

*Picture from [Rempe-Sixsmith, '17]

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Theorem C

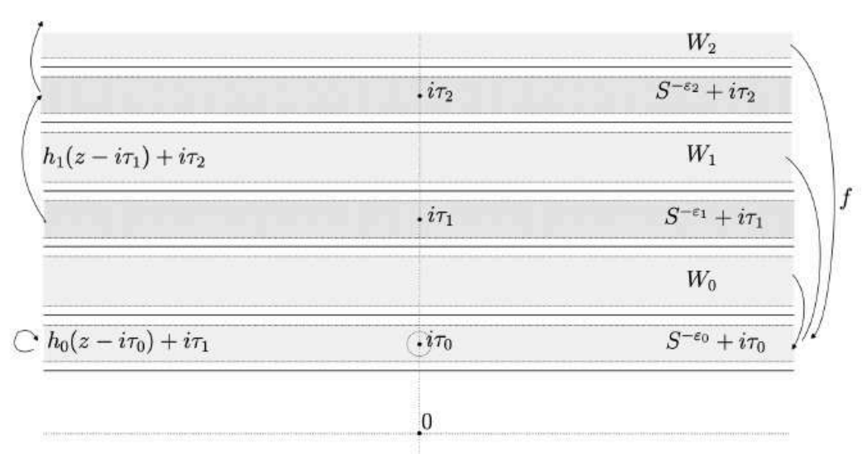
For every $\varepsilon \in (0, 1/2]$, there exists f of order $\rho(f) = \frac{1}{2} + \varepsilon$ with an unbounded fast escaping wandering domain.

IDEA OF PROOFS OF THEOREMS B AND C

Theorem A

There exists an entire function with an orbit of unbounded fast escaping wandering domains.

We start by *designing* our (wandering) sets:



We will approximate our *model map* h.

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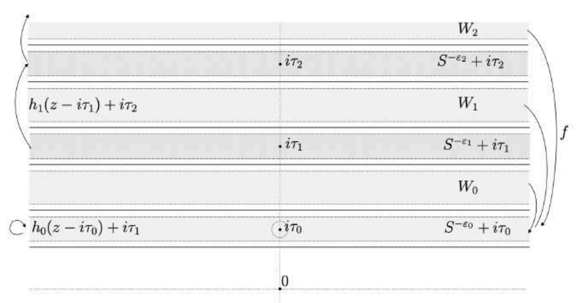

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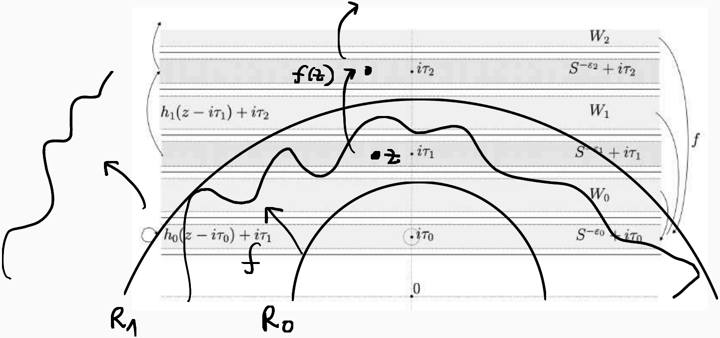

In particular, $\{\tau_k\}$ can be chosen so that f has fast escaping wandering domains.

Hörmander's theorem (*the version we need*)

Let $u : \mathbb{C} \to \mathbb{R}$ be a subharmonic function. Then, for every locally integrable function g there is a solution α of the equation $\bar{\partial}\alpha = g$ such that

$$\int_{\mathbb{C}} |\alpha(z)|^2 \frac{e^{-u(z)}}{(1+|z|^2)^2} dm(z) \leq \frac{1}{2} \int_{\mathbb{C}} |g(z)|^2 e^{-u(z)} dm(z),$$

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- The function $f(z) \coloneqq \chi(z) \cdot h(z) - \alpha(z)$ is entire.

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- Growth: everywhere, $|f(z)| \approx C_2 \cdot |h(z)| + C_3 \cdot \exp(u(z))$.

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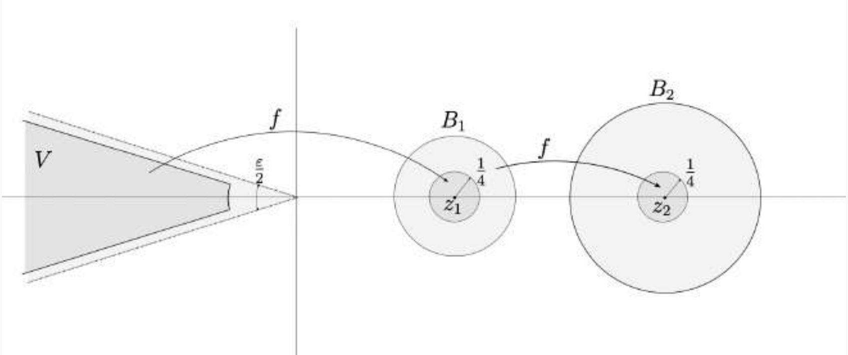
- $f^n(V) \subset B_n$ for all $n \geq 1$.
- V is contained in a (fast escaping) wandering domain.
- $u(z) \leq |z|^{1/2+\varepsilon}$.



gràcies per la vostra atenció!