On the local Birkhoff conjecture
(and other questions in billiard dynamics)

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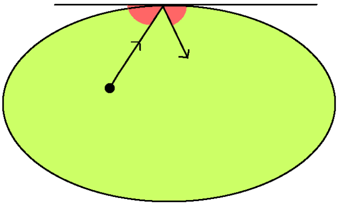
Mathematical Billiards

A mathematical billiard consists of a closed region in the plane (the *billiard table*) and a point-mass in its interior (the *ball*), which moves along straight lines with constant velocity.

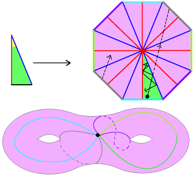
When the ball hits the boundary, it reflects *elastically*, namely:

angle of incidence = angle of reflection.

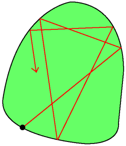


In the case of a table lying in a Riemannian manifold, the ball moves along geodesics instead of straight lines.

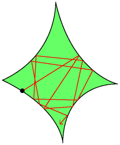
The study of the dynamics of billiards is a very active area of research. Dynamical behaviours and properties are strongly related to the shape of the table.



Polygonal billiards:
- Related to the study of the geodesic flow on a translation surface (with singular points);
- Teichmüller theory.



(Strictly) Convex Billiards:
- Birkhoff billiards (G. Birkhoff, 1927: a paradigm of Hamiltonian systems).
- The billiard map is a twist map.
- Coexistence of regular (KAM, Aubry-Mather) and chaotic dynamics.



Concave Billiards (or dispersive):
- Nearby Orbits tend to move apart (exponentially).
- Hyperbolicity and chaotic behaviour (Y. Sinai, 1970).
- Study of statistical properties of orbits.

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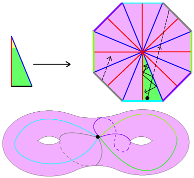

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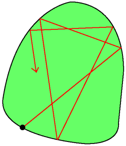

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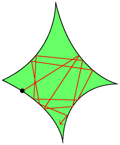

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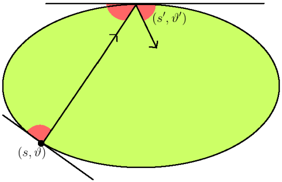
Birkhoff Billiards

Let Ω be a strictly convex domain in $\mathbb{R}^2$ with C^r boundary $\partial\Omega$, with $r \geq 3$. Let $\partial\Omega$ be parametrized by arc-length s (fix an orientation and denote by ℓ its length) and ϑ "shooting" angle (w.r.t. the positive tangent to $\partial\Omega$). The Billiard map is:

$$
\begin{aligned}
B : \mathbb{R}/\ell\mathbb{Z} \times (0,\pi) &\longrightarrow \mathbb{R}/\ell\mathbb{Z} \times (0,\pi) \\
(s,\vartheta) &\longmapsto (s',\vartheta').
\end{aligned}
$$



This simple model has been first proposed by G.D. Birkhoff (1927) as a mathematical playground where "*the formal side, usually so formidable in dynamics, almost completely disappears and only the interesting qualitative questions need to be considered*".

Properties of Birkhoff billiard maps

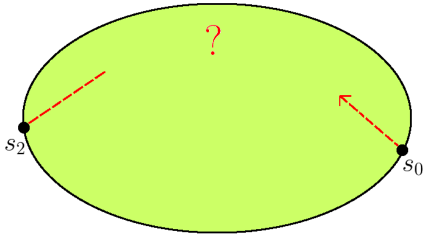
- B is $C^{r-1}(\mathbb{R}/\ell\mathbb{Z} \times (0, \pi))$;
- B can be extended continuously up to the boundary: $B(\cdot, 0) = B(\cdot, \pi) = Id$;
- B preserves the area form $\omega = \sin\vartheta \, d\vartheta \wedge ds$ (symplectic form);
- B is a twist map $\leftarrow$ (Aubry-Mather theory, KAM theory, etc.);
- B has a generating function:

$$h(s, s') := \|\gamma(s) - \gamma(s')\|,$$

i.e., the Euclidean distance between two points on $\partial\Omega$. In particular if $B(s, \vartheta) = (s', \vartheta')$, then:

$$\begin{cases} \partial_1 h(s, s') = -\cos\vartheta \\ \partial_2 h(s, s') = \cos\vartheta' . \end{cases}$$

Dynamics and Length



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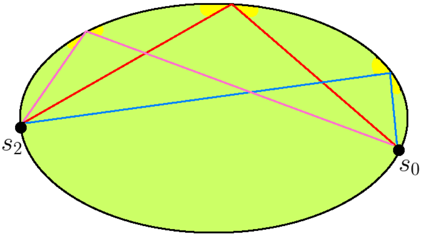

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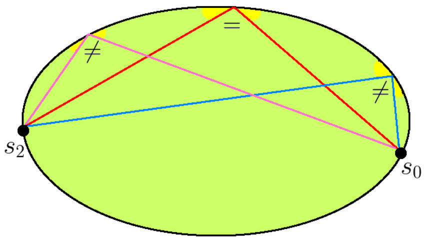

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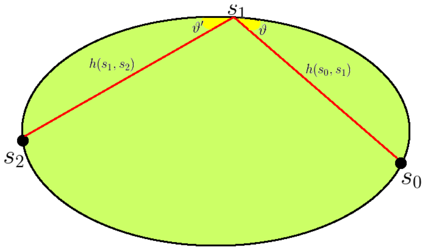


Let us consider the length functional:

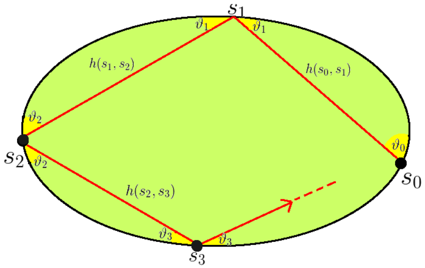
$$\mathcal{L}(s_1) := h(s_0, s_1) + h(s_1, s_2) \quad s_1 \in (s_0, s_2).$$

Then:

$$\frac{d}{ds}\mathcal{L}(s_1) \;=\; \partial_2 h(s_0, s_1) + \partial_1 h(s_1, s_2) = \cos\vartheta - \cos\vartheta'.$$

The real orbit (i.e., $\vartheta = \vartheta'$) correspond to $s_1 \in (s_0, s_2)$ such that $\frac{d}{ds}\mathcal{L}(s_1) = 0$ (i.e., s_1 is a critical point).

Dynamics and Length



$\{(s_n, \vartheta_n)\}_{n\in\mathbb{Z}}$ is an orbit $\iff$ $\{s_n\}_{n\in\mathbb{Z}}$ is a "critical configuration" of the Length functional:

$$\mathcal{L}(\{s_n\}_n) := \sum_{n\in\mathbb{Z}} h(s_n, s_{n+1}).$$

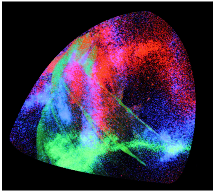
Relation between the Dynamics and the length of trajectories (Geometry).

Dynamics $\longleftrightarrow$ Geometry

Study of Dynamics: understand the properties of orbits (periodicity, symmetries, chaos, etc...)

While the dependence of the dynamics on the geometry of the domain is well perceptible, an intriguing challenge is:

To what extent dynamical information can be used
to reconstruct the shape of the domain.



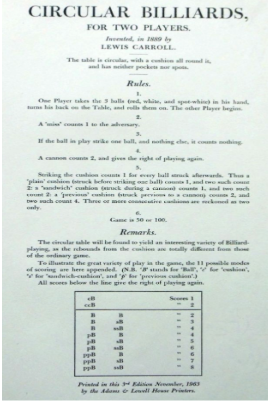
This apparently naïve question is at the core of different intriguing conjectures, among the most difficult to tackle in the study of dynamical systems!

Example I: Circular billiard

Digression: A Mad Tea-Party

Charles Lutwidge Dodgson (1832-1898)
(better known as Lewis Carroll).

'But I don't want to go among mad people', Alice remarked. 'Oh, you can't help that', said the Cat: 'we're all mad here. You're mad.' 'How do you know I'm mad?', said Alice. 'You must be', said the Cat, 'or you wouldn't have come here.'



CIRCULAR BILLIARDS,

FOR TWO PLAYERS.

Invented, in 1889 by
LEWIS CARROLL.

The table is circular, with a cushion all round it,
and has neither pockets nor spots.

Rules.

1.

One Player takes the 3 balls (red, white, and spot-white) in his hand, turns his back on the Table, and rolls them on. The other Player begins.

2.

A 'miss' counts 1 to the adversary.

3.

If the ball in play strike one ball, and nothing else, it counts nothing.

4.

A cannon counts 2, and gives the right of playing again.

5.

Striking the cushion counts 1 for every ball struck afterwards. Thus a 'plain' cushion (struck before striking one ball) counts 1, and two such count 2: a 'sandwich' cushion (struck during a cannon) counts 1, and two such count 2: a 'previous' cushion (struck previous to a cannon) counts 2, and two such count 4. Three or more consecutive cushions are reckoned as two only.

6.

Game is 50 or 100.

Remarks.

The circular table will be found to yield an interesting variety of Billiard-playing, as the rebounds from the cushion are totally different from those of the ordinary game.

To illustrate the great variety of play in the game, the 11 possible modes of scoring are here appended. (N.B. '*B*' stands for 'Ball', '*c*' for 'cushion', '*s*' for 'sandwich-cushion', and '*p*' for 'previous cushion'.)

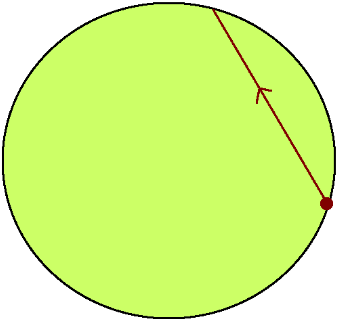
All scores below the line give the right of playing again.

cB		Scores	1
ccB		"	2
B	B	"	2
B	sB	"	3
B	ssB	"	4
pB	B	"	4
pB	sB	"	5
pB	ssB	"	6
ppB	B	"	6
ppB	sB	"	7
ppB	ssB	"	8

Printed in this 3rd Edition November, 1963
by the Adams & Lowell House Printers.

Lewis Carroll thought of playing billiards on a circular table in 1889 and first published its rules the following year (and a circular billiard table was actually made for him!)

Example I: Circular billiard



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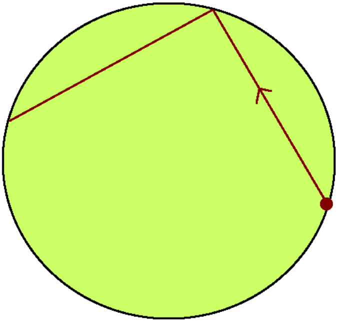

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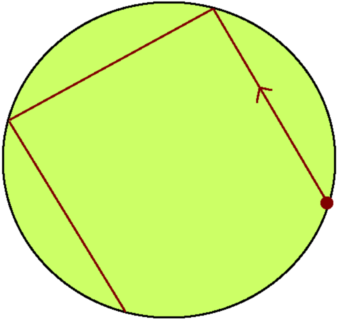

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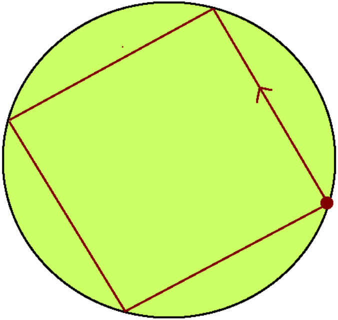

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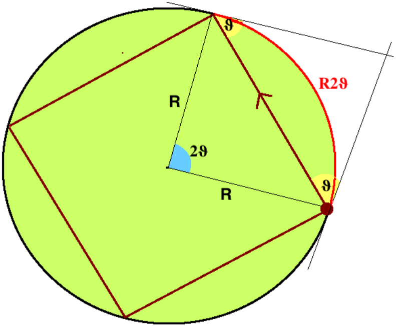
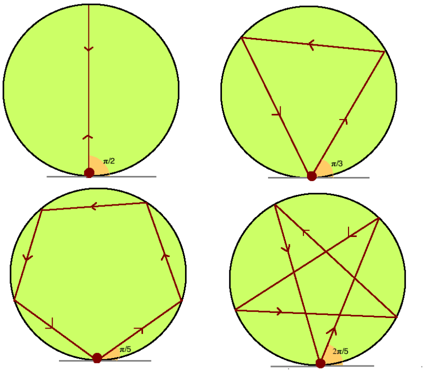


The angle remains constant at each bounce: it is an Integral of motion.
This is an example of integrable dynamical system.

Example I: Circular billiard

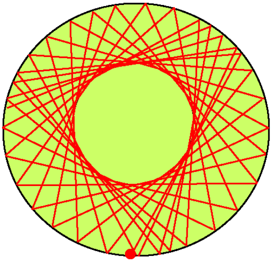
If ϑ is a rational multiple of π, then the resulting orbit is periodic:



For every rational $\frac{p}{q} \in (0, \frac{1}{2}]$ there exist infinitely many periodic orbits with q bounces (period) and which turn p times around before closing (winding number). $\frac{p}{q}$ is called rotation number.

Example I: Circular billiard

If ϑ is NOT a rational multiple of π, then the orbit hits the boundary on a dense set of points (Kroenecker's theorem):



The trajectory does not fill in the table: there is a region (a disc) which is never crossed by the ball!

Observe that the trajectory is always tangent to a circle (this is an example of caustic).

What is true for general Birkhoff billiards?

- Do there always exist periodic orbits? How many?

- How often does the existence of caustics occur? Are there other integrable billiards?

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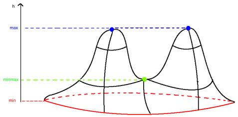
  YES! For every rotation number $\frac{p}{q} \in (0, \frac{1}{2}]$ there exist at least two distinct periodic orbits with that rotation number (Birkhoff, 1922).

A variation proof exploits the relation between orbits and lengths: one of the two orbits maximizes the length among all configurations with that rotation number, while the other is obtained via a min-max procedure.



(Mountain pass lemma)

Q1 - Do the collection of their lenghts encode any information on Ω?

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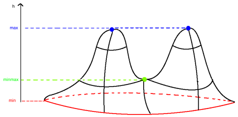

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Q1 - Do the collection of their lenghts encode any information on Ω?

- How often does the existence of caustics occur? Are there other integrable billiards? $\longrightarrow$ Birkhoff conjecture

  Q2 - What does integrability say about the geometry of the table?

Integrability of billiards

There are several ways to define integrability for Hamiltonian systems:

- Liouville-Arnol'd integrability (existence of integrals of motion);
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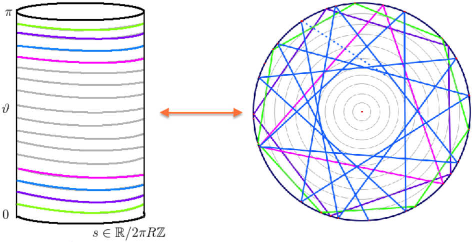
Is it possible to express the integrability of a billiard map in terms of property of the billiard table?



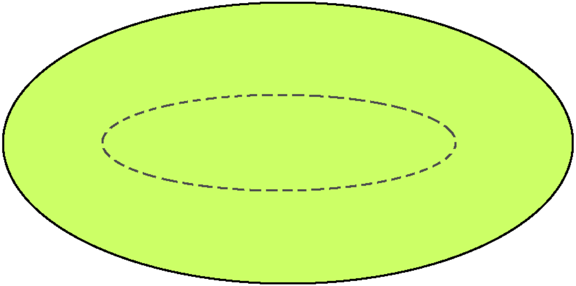
Phase space of the circular billiard map

Foliation of the circular billiard table by caustics

Integrability $\longleftrightarrow$ (Part of) the billiard table is foliated by caustics

Caustics

A convex caustic is a closed C^1 curve in the interior of Ω, bounding itself a strictly convex domain, with the property that each trajectory that is tangent to it stays tangent after each reflection.



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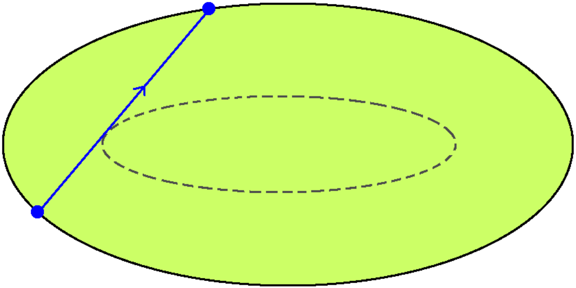

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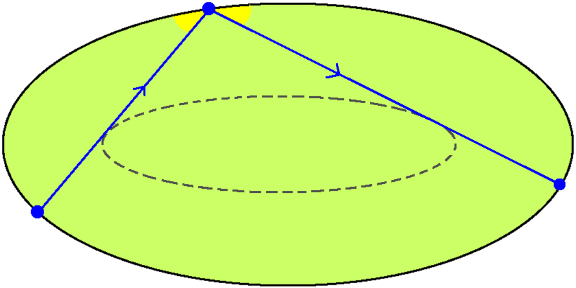

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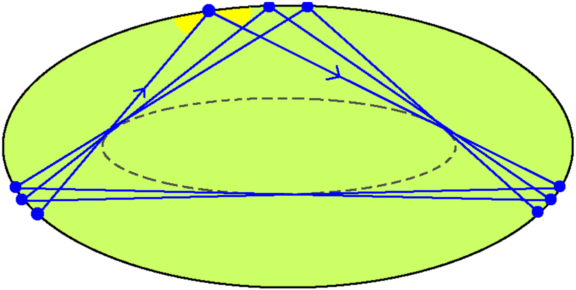

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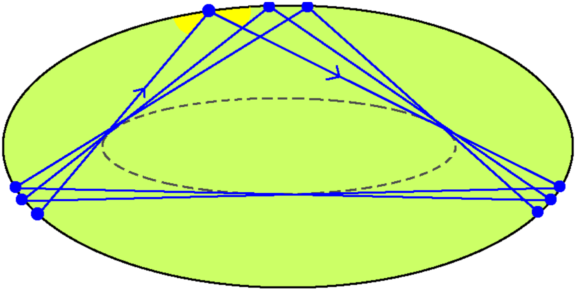
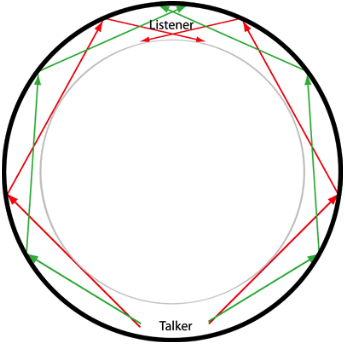


To a convex caustic in Ω corresponds an invariant circle for the billiard map.

(The converse is not entirely true: invariant curves give rise to caustics, but they might not be convex, nor differentiable).

Digression: Caustics and Whispering Galleries

Whispering Gallery in St. Paul Cathedral in London (Lord Rayleigh, 1878 ca.)



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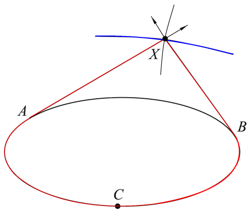

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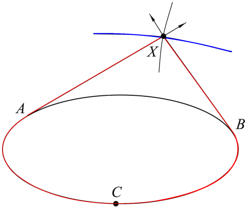

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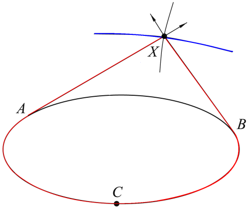

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  YES! Lazutkin (1973) proved that by a suitable change of coordinates every Birkhoff billiard map becomes nearly integrable!
  Hence, if the domain is sufficiently smooth, he proved by means of KAM technique that there exists (at least) a Cantor set of invariant circles near the boundary (i.e., infinitely many caustics accumulating to the boundary of the table).

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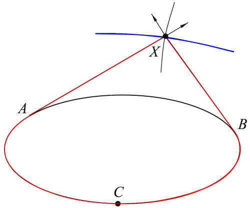

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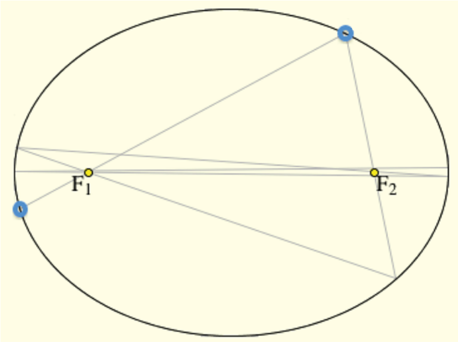
- Do there exist other examples of billiards admitting a foliation by caustics?

Example II: Elliptic billiard

Curiosity: The New York Times (1st July 1964) ran a full-page ad for Elliptipool, played on an elliptical table with a single pocket at one of the two foci. The ad said that on the following day the game would be demonstrated at Stern's department store by movie stars Paul Newman and Joanne Woodward.

Example II: Elliptic billiard



If the trajectory passes through one of the foci, then it always passes through them, alternatively.

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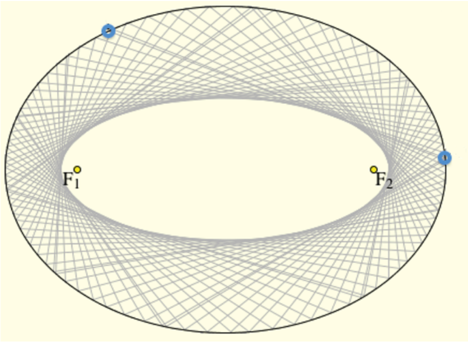

If the trajectory does not intersect the segment between the foci, then it never does and it is tangent to a confocal ellipse (a convex caustic).

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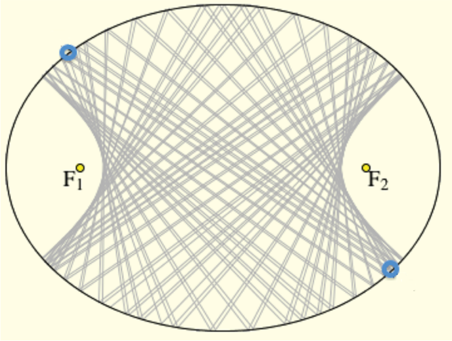

If the trajectory intersects the segment between the foci, then it always does and it is tangent to a confocal hyperbola (a non-convex caustic).

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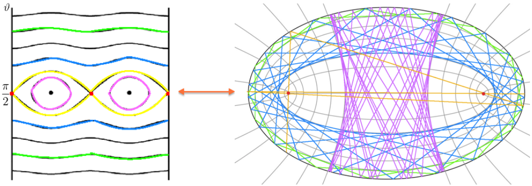


Phase space of an elliptic billiard map

Dynamics inside an elliptic billiard and caustics

Some Properties of Elliptic billiards:

- For every rational $\frac{p}{q} \in (0, \frac{1}{2})$ there exist infinitely many periodic orbits rotation number $\frac{p}{q}$.
- There exist only two periodic orbits of period 2 (i.e., rotation number $\frac{1}{2}$): the two semi-axes.
- There exist infinitely many convex caustics (and also non-convex ones).

The ellipse, with the exception of the closed segment between the foci, is foliated by convex caustics. It is an Integrable billiard.

Birkhoff conjecture

Conjecture (Birkhoff-Poritsky)

The only integrable billiard maps correspond to billiards inside ellipses.

Although some vague indications of this question can be found in Birkhoff's works (1920's-30's), its first appearance was in a paper by Poritsky (1950), who was a National Research Fellow in Mathematics at Harvard University, presumably under the supervision of Birkhoff.

It quickly became one of the most famous - and hard to tackle - questions in dynamical systems.

It is important to consider strictly convex domains!
Mather (1982) proved the non-existence of caustics (hence, some sort of non-integrability) if the curvature of the boundary vanishes at (at least) one point. See also Gutkin-Katok (1995).

Previous contributions

Despite its long history and the amount of attention that this conjecture has captured, it remains still open. Important contributions are the following:

- Bialy (1993): If the phase space of the billiard map is completely foliated by continuous invariant curves which are not null-homotopic, then it is a circular billiard. An integral-geometric approach to prove Bialy's result was proposed by Wojtkowski (1994), by means of the so-called mirror formula.
- Innami (2002) showed that the existence of caustics with rotation numbers accumulating to $1/2$ implies that the billiard is an ellipse; the proof is based on Aubry-Mather theory (a simpler proof by Arnold-Bialy (2018)).
- In a different setting, when there exists an integral of motion that is polynomial in the velocity (Algebraic Birkhoff conjecture), the fact that the billiard is an ellipse has been recently proved thanks to the contributions of Glutsyuk (2018) and Bialy-Mironov (2017).

Perturbative Birkhoff conjecture

One could restrict the analysis to what happens for domains that are sufficiently close to ellipses.

Birkhoff Conjecture (Perturbative version)

A smooth strictly convex domain that is sufficiently close (w.r.t. some topology) to an ellipse and whose corresponding billiard map is integrable, is necessarily an ellipse.

- First results in this direction were obtained by:
  - Levallois (1993): Non-integrability of algebraic perturbations of elliptic billiards.
  - Delshams and Ramírez-Ros (1996): Non-integrability of entire symmetric perturbations of ellipses (these perturbations break integrability near the homoclinic solutions).
- Avila, De Simoi and Kaloshin (2016) proved that perturbative version of Birkhoff conjecture holds true for domains that are nearly circular.

Main Result: the Perturbative Birkhoff Conjecture

Our main result is that the Perturbative Birkhoff conjecture holds true for any ellipse. More specifically:

Theorem [Kaloshin - S. (*Annals of Math.*, 2018)]

Let $\mathcal{E}_0$ be an ellipse of eccentricity $0 \leq e_0 < 1$ and semi-focal distance c; let $k \geq 39$. For every $K > 0$, there exists $\varepsilon = \varepsilon(e_0, c, K)$ such that the following holds.
Let Ω be a C^k domain such that:

- Ω admits integrable rational caustics$^{(*)}$ of rotation number $1/q$, $\forall\, q \geq 3$,
- $\partial\Omega$ is K-close to $\mathcal{E}_0$, with respect to the C^k-norm,
- $\partial\Omega$ is ε-close to $\mathcal{E}_0$, with respect to the C^1-norm,

then Ω must be an ellipse.

(*) An integrable rational caustic corresponds to a (non-contractible) invariant curve of the billiard map foliated by periodic points.

Local integrability and Birkhoff conjecture

One could consider weaker notions of integrability.

For example: what can be said for locally integrable Birkhoff billiards? Namely, consider integrability in a neighborhood of the boundary of the billiard table, i.e., for sufficiently small rotation numbers.

The analogous conjecture would be:

Local Birkhoff Conjecture (LBC)

If Ω is a Birkhoff billiard admitting a foliation by caustics with rotation numbers in $(0, \delta)$, for some $0 < \delta \leq 1/2$, then Ω must be an ellipse.

For $\delta = 1/2$ it follows from the result by Innami (2002).

Local Perturbative Birkhoff conjecture (LPBC)

Theorem [Huang, Kaloshin, S. (*GAFA*, 2018)]

For any integer $q_0 \geq 2$, there exist $e_0 = e_0(q_0) \in (0, 1)$, $m_0 = m_0(q_0)$, $n_0 = n_0(q_0) \in \mathbb{N}$ such that the following holds.

For each $0 < e \leq e_0$ and $c \geq 0$, there exists $\varepsilon = \varepsilon(e, c, q_0) > 0$ such that if

- $\mathcal{E}_0$ is an ellipse of eccentricity e and semi-focal distance c,
- Ω admits integrable rational caustics for all $0 < \frac{p}{q} \leq \frac{1}{q_0}$,
- $\partial\Omega$ is C^{m_0} domain,
- $\partial\Omega$ is ε-close (in the C^{n_0} topology) to $\mathcal{E}_0$,

$\Longrightarrow$ Ω itself is an ellipse.

- For $q_0 = 2, 3, 4, 5$, we have $m_0 = 40q_0$ and $n_0 = 3q_0$.
- For $q_0 > 5$, we have $m_0 = 40q_0$ and $n_0 = 3q_0$, BUT the result is subject to checking that $q_0 - 2$ matrices (which are explicitely described) are invertible.
- A complete proof for all $q_0 > 5$ has been provided by Ilya Koval (2021).

A recent result by Bialy & Mironov

Recently, Bialy-Mironov (*Annals of Math.*, 2022) proved a version of Birkhoff conjectures for centrally-symmetric C^2 Birkoff billiards, under the assumption that a neighborhood of the boundary has a C^1-smooth foliation by caustics of rotation numbers in $(0, 1/4]$. (See also an effective version by Bialy-Tsodikovich, 2022).

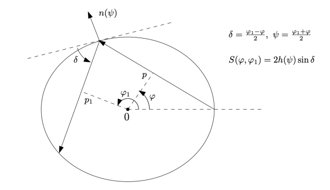
The main ingredients are:

- a non-standard generating function (Bialy-Mironov, *Adv. Math.* 2017);
- the remarkable structure of the invariant curve consisting of 4-periodic orbits (for centrally symmetric domains, they are parallelograms as for ellipses);
- the integral-geometry approach by Bialy (*Math. Z.*, 1993) for rigidity results of circular billiards, here used to establish a Hopf-type rigidity for billiards in ellipses.

Fix an origin in $\mathbb{R}^2$ and consider coordinates (p, φ) on the space of all oriented lines (see figure).
The symplectic form is $d\beta$ with $\beta = p d\varphi$.

Let h be the support function of the boundary w.r.t. origin, *i.e.* the boundaries of the phase cylinder are $\{p = h(\varphi)\}$, $\{p = -h(\varphi + \pi)\}$.

The generating function w.t.r. to β is
$S(\varphi, \varphi_1) = 2h(\frac{\varphi+\varphi_1}{2}) \sin(\frac{\varphi_1-\varphi}{2})$.



Integrable geodesic flows on the Torus

Birkhoff conjecture can be thought as the analogue, in the case of billiards, of the following question: classify integrable (Riemannian) geodesic flows on $\mathbb{T}^2$.

The complexity of this question depends on the notion of integrability.

- If one assumes that the whole space space is foliated by invariant Lagrangian graphs (C^0-integrability), then it follows from Hopf conjecture that the associated metric must be flat. (Similar to Bialy's result for billiards.)
- This question is still open if one considers integrability only on an open and dense set (global integrability), or assumes the existence of an open set foliated by invariant Lagrangian graphs (local integrability).

Example of globally integrable (non-flat) geodesic flows on $\mathbb{T}^2$ are those associated to Liouville-type metrics:

$$ds^2 = (f_1(x_1) + f_2(x_2))\,(dx_1^2 + dx_2^2).$$

Folklore conjecture: these metrics are the only globally (resp. locally) integrable metrics on $\mathbb{T}^2$.

Periodic orbits and the (Marked) Length spectrum

What can of information does the set of periodic orbits encode?

We define the Length spectrum of Ω:

$$\mathcal{L}(\Omega) := \mathbb{N}^+ \cdot \{\text{lengths of billiard periodic orbits in } \Omega\} \cup \ell \cdot \mathbb{N}^+.$$

One could also refine $\mathcal{L}(\Omega)$. Consider pairs (length, rotation number) and define the Marked Length spectrum $\mathcal{ML}_\Omega$.
In particular, for every $p/q \in (0, 1/2]$ define:

$$\mathcal{ML}_\Omega(p/q) := \max\{\text{lenghts of per. orbits of rot. number } p/q\}.$$

This is also related to Mather's β-function for billiards:

$$\beta(p/q) := -\frac{1}{q}\mathcal{ML}_\Omega(p/q).$$

From the spectrum to the dynamics

What dynamical information does $\mathcal{ML}_\Omega$ encode?

Theorem [Huang, Kaloshin, S. (*Duke Math. J.*, 2018)]

For (Baire) generic billiard domain, it is possible to recover from the (maximal) marked length spectrum, the Lyapunov exponents of its Aubry-Mather (A-M) orbits), i.e., the periodic orbits with maximal length in their rotation number class.

IDEA: Approximate an A-M orbit by a suitable sequence of other A-M orbits, do an asymptotic analysis of their minimal averaged action and show that this allows to recover its Lyapunov exponents....

From the spectrum to the dynamics

More precisely, for a generic strictly convex $C^{\tau+1}$-billiard table Ω ($\tau \geqslant 2$), we have that for each $p/q \in \mathbb{Q} \cap (0, 1/2]$ in lowest terms:

- The following limit exists

$$\lim_{N\to+\infty} \left[\mathcal{ML}_\Omega\left(\frac{Np}{Nq-1}\right) - N \cdot \mathcal{ML}_\Omega\left(\frac{p}{q}\right)\right] = -B_{p/q}$$

  where $B_{p/q}$ denotes the minimum value of Peierls' Barrier function of rotation number p/q.

- Moreover:

$$\lim_{N\to+\infty} \frac{1}{N} \log \left| \mathcal{ML}_\Omega(\frac{Np}{Nq-1}) - N \cdot \mathcal{ML}_\Omega(\frac{p}{q}) + B_{p/q} \right| = \log \lambda_{p/q}$$

  where $\lambda_{p/q}$ is the eigenvalue of the linearization of the Poincaré return map at the Aubry-Mather periodic orbit with rotation number $\frac{p}{q}$.

Can you hear the shape of a drum?

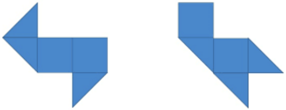
Let $\Omega \subset \mathbb{R}^2$ and consider the problem of finding $u \not\equiv 0$ and $\lambda \in [0, +\infty)$ such that:

$$\begin{cases} \Delta u + \lambda^2 u = 0 & \text{in } \Omega \\ u = 0 & \text{on } \partial\Omega. \end{cases}$$

We define the Laplace Spectrum as: $\mathrm{Spec}(\Omega) := \{0 < \lambda_1 \leq \lambda_2 \leq \ldots\}$.

Kac's question (1966): Does $\mathrm{Spec}(\Omega)$ determine Ω up to isometry?

- The answer is well-known to be negative (all known examples are not convex and they are bounded by curves that are only piecewise analytic).
- (Osgood-Phillips-Sarnak) A C^∞ isospectral set is compact.
- (Zelditch, *Ann. of Math.* 2009) positive answer for generic analytic axial-symmetric convex domains.
- (Hezari-Zelditch, *Ann. of Math.* 2022) positive answer for ellipses of small eccentricities (spectrally determined among all smooth domains).



Counterexample by

Gordon-Webb-Wolpert (1992)

Laplace Spectrum and Length Spectrum

There is a deep relation between the Laplace spectrum and the Length spectrum.

Theorem (Andersson and Melrose, *Invent. Math.* 1977)

The wave trace $w(t) := \mathrm{Re}\left(\sum_{\lambda_n \in \mathrm{Spec}(\Omega)} e^{i\lambda_n t}\right)$ is well-defined as a distribution and it is smooth away from the length spectrum:

$$\text{sing. supp.}\big(w(t)\big) \subseteq \pm\mathcal{L}(\Omega) \cup \{0\}.$$

Generically, equality holds.

Hence, at least for generic domains, one can recover the length spectrum from the Laplace one.

Can you hear the shape of a billiard?

Question: If $\mathcal{L}(\Omega_1) \equiv \mathcal{L}(\Omega_2)$, or alternatively $\mathcal{ML}_{\Omega_1} \equiv \mathcal{ML}_{\Omega_2}$, is it true that Ω_1 and Ω_2 must be isometric?

In Riemannian geometry, similar questions have been studied in the case of negatively curved surfaces (Guillemin, Kazhdan,Croke, Otal, Fathi, etc...) and some higher dimensional case (Guillarmou-Lefeuvre, 2019).

Affirmative answer if one of the two is a disc (easy).

What about ellipses?

Theorem [Kaloshin, S.]

If a domain is "close" to an ellipse and has the same Marked Length spectrum of an ellipse, then it must be an ellipse .

Spectral rigidity of convex billiards

Ω is called spectrally rigid is any C^1-smooth one-parameter isospectral family $\{\Omega_\varepsilon\}_{|\varepsilon|<\varepsilon_0}$ with $\Omega_0 = \Omega$ is necessarily an isometric family.

Question: Are Birkhoff billiards spectrally rigid?

De Simoi, Kaloshin and Wei (*Ann. of Math.* 2016) proved that this is true for almost circular strictly convex domains, axial symmetric and with sufficiently smooth boundary.

What about domains close to ellipses of any eccentricity or generic domains?

(Marked) Action spectrum of a Tonelli Hamiltonian system

Similar questions are meaningful for Hamiltonian systems and twist maps.

Definition (Tonelli Hamiltonian)

Let M be finite dimensional closed Riemannian manifold. $H \in C^2(T^*M, \mathbb{R})$ is said to be Tonelli if:

- H is strictly convex in each fibre: $\partial^2_{pp} H(x,p) > 0$;
- H is superlinear in each fibre: $\lim_{\|p\| \to +\infty} \frac{H(x,p)}{\|p\|} = +\infty$ uniformly in x.

One can associate to a Tonelli Hamiltonian two functions (Mather's minimal actions):

$$\alpha : H^1(M,\mathbb{R}) \longrightarrow \mathbb{R} \qquad \text{and} \qquad \beta : H^1(M,\mathbb{R}) \longrightarrow \mathbb{R}$$

which are convex, superlinear and in convex duality.

- α coincides with the symplectic homogenization of H.

What dynamical information are encoded in these functions?

Some results on $\mathbb{T}^2$:

- Let $h_0 \in H_1(\mathbb{T}^2;\mathbb{R})$ be 1-irrational (i.e., on a line with rational slope). β is differentiable at h_0 if and only if there exists an invariant Lagrangian torus foliated by periodic orbits with rotation vector h_0 (Massart-S., 2011).

  (Analogue result for twist maps of the 1-dimensional annulus by Mather and Bangert).

- $\beta \in C^1(H_1(\mathbb{T}^2,\mathbb{R}))$ if and only if H is C^0-integrable (i.e., the phase space is foliated by invariant Lipschitz Lagrangian graph, one for each cohomology class) (Massart-S., 2011).

Some open questions

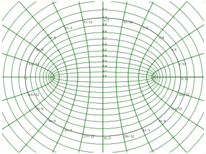
- What happens on $\mathbb{T}^n$ with $n > 2$?
- If two Hamiltonian systems have the same β or α functions, are their dynamics related?
- What dynamical information do the β and α function encode?

Gràcies per la vostra atenció
i pel premi:
estem realment honrats de rebre-ho!

V. Kaloshin, A. Sorrentino, J. Mather (Lyon, 2012)

Sketch of the proof of Theorem [Kaloshin-S.] 1/5

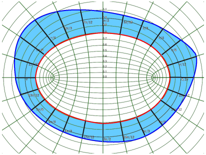
- Consider elliptic coordinates (μ, φ):



$$\begin{cases} x = c\cosh\mu\,\cos\varphi \\ y = c\sinh\mu\,\sin\varphi \end{cases}$$

describing confocal ellipses ($\mu = \mu_0$) and hyperbolae ($\varphi = \varphi_0$); $c > 0$ represents the semifocal distance.

- We express a perturbation of a given ellipse $\{\mu_{=}\mu_0\}$ as:



$$\mu_\varepsilon(\varphi) = \mu_0 + \varepsilon\mu_1(\varphi) + O(\varepsilon^2).$$

(Observe that the coordinate frame depends on the unperturbed ellipse)

Sketch of the proof of Theorem [Kaloshin-S.] 2/5

Let us start by considering a rationally integrable deformation Ω_ε of $\Omega_0 = \mathcal{E}_0$.

Action-angle coordinates for the billiard map in the ellipse $\mathcal{E}_0$. For $q \geq 3$, let $\varphi_q(\theta)$, $\theta \in \mathbb{R}/2\pi\mathbb{Z}$, denote the parametrization of the boundary induced by the dynamics on the caustic of rotation number $1/q$:

$$B_{\mathcal{E}_0}(\mu_0, \varphi_q(\theta)) = (\mu_0, \varphi_q(\theta + 2\pi/q)).$$

Lemma [Pinto-de-Carvalho, Ramírez-Ros (2013)]

Let Ω_ε admit a rationally integrable caustic of rotation number $1/q$ for all ε. We denote by $\{\varphi_q^k\}_{k=0}^q$ the periodic orbit of the billiard map in $\mathcal{E}_0$ with rotation number $1/q$ and starting at φ; then $L_1(\varphi) = \sum_{k=1}^q \mu_1(\varphi_q^k) \equiv c_q$, where c_q is a constant independent of φ.

$L_1(\varphi)$ represents the subharmonic Melnikov potential of the elliptic caustic of rotation number $1/q$ under the deformation.

Sketch of the proof of Theorem [Kaloshin-S.] 3/5

Therefore, with respect to the action-angle variables we have that for any $\theta \in \mathbb{R}/2\pi\mathbb{Z}$:

$$\sum_{k=1}^{q} \mu_1(\varphi_q(\theta + 2\pi k/q)) \equiv \mathrm{c_q}.$$

If $u(x)$ denotes either $\cos x$ and $\sin x$, then

$$\int_0^{2\pi} \mu_1(\varphi_q(\theta))\, u(q\,\theta)\, d\theta = 0,$$

which, using the expression for φ_q and by some change of variables, implies:

$$\int_0^{2\pi} \mu_1(\varphi)\, \frac{u\left(\frac{2\pi\, q}{4K(k_q)} F(\varphi; k_q)\right)}{\sqrt{1 - k_q^2 \sin^2\varphi}}\, d\varphi = 0.$$

- k_q is the eccentricity of the elliptic caustic of rotation number $1/q$
- $F(\varphi, k)$ the incomplete elliptic integral of the first kind;
- $K(k)$ the complete elliptic integral of the first kind, i.e. $K(k) = F(\pi/2, k)$.

Sketch of the proof of Theorem [Kaloshin-S.] 4/5

We define a family of dynamical modes $\{c_q, s_q\}_{q\geq 3}$ given by

$$c_q(\varphi) := \frac{\cos\left(\frac{2\pi\, q}{4K(k_q)} F(\varphi; k_q)\right)}{\sqrt{1 - k_q^2 \sin^2\varphi}} \qquad s_q(\varphi) := \frac{\sin\left(\frac{2\pi\, q}{4K(k_q)} F(\varphi; k_q)\right)}{\sqrt{1 - k_q^2 \sin^2\varphi}}.$$

These functions only depend on μ_0 and q.

Summarizing: if $\mu_\varepsilon(\varphi) = \mu_0 + \varepsilon\mu_1(\varphi) + O(\varepsilon^2)$ is a deformation of the ellipse $\mathcal{E}_0 = \{\mu = \mu_0\}$ which preserves the integrable caustic of rotation number $1/q$, then

$$< \mu_1, c_q >_{L^2} = < \mu_1, s_q >_{L^2} = 0$$

Consider also five extra functions related to elliptic motions: $e_1, \ldots, e_5$: they correspond to infinitesimal generators of motions that transform ellipses into ellipses (translations, rotations, homotheties, hyperbolic rotations).

Sketch of the proof of Theorem [Kaloshin-S.] 5/5

Key result: Basis property

$\{e_j\}_{j=1}^5 \cup \{c_q, s_q\}_{q\geq 3}$ form a basis of $L^2(\mathbb{T})$.

Idea: Make them (more) complex!

- Consider complex analytic extensions of these functions.
- A detailed study of their complex singularities and the size of their maximal strips of analiticity, allow us to deduce their linear independence (both for finite and infinite combinations).
- By a codimension argument, show that they form a set of generators.

From Deformative to Perturbative Setting:

- Annihilation conditions are replaced by smallness condition;
- Approximate $\partial\Omega$ with its "best" approximating ellipse:

$$\partial\Omega = \{(\mu_0^* + \mu_{\mathrm{pert}}(\varphi), \varphi):\ \varphi \in [0, 2\pi)\};$$

- Using smallness conditions and Basis property, deduce that $\|\mu_{\mathrm{pert}}\|_{L^2}$ must be zero.