Sistemes dinàmics no autònoms: alguns resultats possiblement sorprenents

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Talk based on the papers:

A. Cima, A. Gasull, V. Mañosa. *Parrondo's dynamic paradox for the stability of non-hyperbolic fixed points*. Discrete Contin. Dyn. Syst. **38** (2018) 889–904.

A. Cima, A. Gasull, V. Mañosa. *Integrability and non-integrability of periodic non-autonomous Lyness recurrences.* Dyn. Syst. **28** (2013) 518–538.

A. Cima, A. Gasull, V. Mañosa. *Non-autonomous 2-periodic Gumovski-Mira difference equations.* Internat. J. Bifur. Chaos Appl. Sci. Engrg., **22** (2012) 1250264 (14 pages).

Outline of the talk

1. Our goal: local and global surprises
2. Local case: simple situations
3. Parrondo's paradox
4. Local case: concrete examples and proofs
5. Global case 1: Lyness recurrences
6. Global case 2: McMillan–Gumovski–Mira recurrences

Outline of the talk

Periodic discrete dynamical systems-I

The study of periodic discrete dynamical systems is a classical topic that has attracted the researcher's interest in the last years, among other reasons, because they are good models for describing the dynamics of biological systems under periodic fluctuations due to external disturbances or effects of seasonality.

These k-periodic systems can be written as

$$x_{n+1} = f_{n+1}(x_n),$$

with initial condition x_0, and a set of maps $\{f_m\}_{m\in\mathbb{N}}$ such that $f_m = f_\ell$ if $m \equiv \ell \pmod{k}$. For short, the set $\{f_1, \ldots, f_k\}$ will be called periodic set.

Periodic discrete dynamical systems-II

It is well-known that given a periodic discrete dynamical system,

$$x_{n+1} = f_{n+1}(x_n),$$

it can be studied via the composition map:

$$f_{k,k-1,\ldots,1} = f_k \circ f_{k-1} \circ \cdots \circ f_1.$$

For instance, if all maps $f_1, f_2, \ldots, f_k$ share a common fixed point p, its nature can be studied through the nature of the fixed point p of

$$f_{k,k-1,\ldots,1}.$$

Questions

LOCAL

If p is a fixed point locally asymptotically stable (LAS) for each map $f_i : \mathbb{R}^m \to \mathbb{R}^m$, $\Rightarrow$ p is **?** for $f_{k,k-1,\ldots,1}$

GLOBAL-1

If f_1 has "complicated" dynamics, can we choose some f_2, with "simple" dynamics, and such that the composition map $f_{2,1}$ has simple dynamics?

GLOBAL-2

If each f_i is conjugated with some g_i, $i = 1, 2$, is there some relation between the dynamics of $f_{2,1}$ and the dynamics of $g_{2,1}$?

Local question

As we will see the answer to our LOCAL question depends on:

- the dimension m,
- the number of maps k,
- the hyperbolicity of the fixed points.

Global question-1

To find examples where our GLOBAL question has a positive answer we consider several maps obtained studying the non-autonomous periodic second order Lyness difference equations

$$x_{n+2} = \frac{a_n + x_{n+1}}{x_n},$$

where $\{a_n\}$ is a cycle of k positive numbers, i.e. $a_{n+k} = a_n$.

Global question-2

To find examples where the dynamics of $f_{2,1}$ and the dynamics of $g_{2,1}$ have NO relation at all we study the non-autonomous two periodic second order McMillan–Gumovski –Mira type difference equations

$$x_{n+2} = -x_n + \frac{a_n x_{n+1}}{1 + x_{n+1}^2},$$

and

$$x_{n+2} = -x_n + \frac{x_{n+1}}{b_n + x_{n+1}^2},$$

where $\{a_n\}$ and $\{b_n\}$ are 2 periodic cycles.

Recall the local question

If p is a fixed point locally asymptotically stable (LAS) for each map $f_i : \mathbb{R}^m \to \mathbb{R}^m$, $\Rightarrow$ p is ? for $f_{k,k-1,\ldots,1}$

Outline of the talk

A known linear planar example

In the book: *When a pair of matrices is stable?*, Blondel, Theys and Tsitsiklis (2004), the authors consider the linear maps $f_i(\mathbf{x}) = A_i \cdot \mathbf{x}^t$, $i = 1, 2$ where

$$A_1 = \alpha \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} \text{ and } A_2 = \alpha \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix},$$

with $|\alpha| < 1$. Both maps have the origin as a GAS point because $\operatorname{Spec}(A_1) = \operatorname{Spec}(A_2) = \{\alpha\}$. Then the composition map is $f_{2,1}(\mathbf{x}) = A_{2,1} \cdot \mathbf{x}^t$ with

$$A_{2,1} = \alpha^2 \begin{pmatrix} 1 & 1 \\ 1 & 2 \end{pmatrix},$$

and it is such that $\operatorname{Spec}(A_{2,1}) = \{ \left(3 \pm \sqrt{5}\right) \alpha^2/2 \}$. Hence the origin is either GAS if $|\alpha| < (\sqrt{5} - 1)/2 \simeq 0.618$, or a saddle point if $(\sqrt{5} - 1)/2 < |\alpha| < 1$.

The stability of the origin is lost.

The hyperbolic case

In the HYPERBOLIC CASE in $\mathbb{R}^m$, for all m,

If p is LAS for each $f_i \Rightarrow p$ is NOT REPELLER for $f_{k,k-1,\ldots,1}$

Moreover when $m = 1$,

If p is LAS for each $f_i \Rightarrow p$ is LAS for $f_{k,k-1,\ldots,1}$

The hyperbolic case-II

PROOF:
Since p is LAS and hyperbolic, for each i,

$$\det\big(Df_i(p)\big) = \lambda_1\lambda_2\cdots\lambda_m < 1.$$

Hence

$$\det\big(Df_{k,k-1,\ldots,1}(p)\big) = \det\big(Df_k(p)\big)\det\big(Df_{k-1}(p)\big)\cdots\det\big(Df_1(p)\big) < 1.$$

In particular, at least one eigenvalue of $Df_{k,k-1,\ldots,1}(p)$ is smaller that 1, and the point p has at least an 1-dimensional stable manifold.

Outline of the talk

Parrondo's paradox

Parrondo's paradox is a paradox in game theory, that essentially says that a combination of losing strategies becomes a winning strategy.

- J.M.R. Parrondo. *How to cheat a bad mathematician.* in EEC HC&M Network on Complexity and Chaos, ISI, Torino, Italy (1996), Unpublished.
- G.P. Harmer and D. Abbott. *Losing strategies can win by Parrondo's paradox.* Nature (London), Vol. 402, No. 6764 (1999) p. 864.

What happens without the hyperbolicity restriction?

In the GENERAL CASE in $\mathbb{R}^m$, is it possible to find $f_1, f_2, \ldots, f_k$ such that:

The point p is LAS for each f_i while p is REPELLER for $f_{k,k-1,\ldots,1}$?

Main results: Parrondo's Paradox-I

DIMENSION 1

Theorem

The following statements hold:

(a) *Consider two analytic maps f_1, f_2, having a common fixed point p which is LAS. Then, the point p is either LAS or semi-AS for the composition map $f_{2,1}$ and both possibilities may happen.*

(b) *There are $k \geq 3$ polynomial maps f_i, $i = 1, 2, \ldots, k$ sharing a common fixed point p which is LAS for all them and such that p is a repeller fixed point for the composition map $f_{k,k-1,\ldots,1}$.*

Main results: Parrondo's Paradox-II

DIMENSION 2

Theorem

There exist polynomial maps f_1 and f_2 in $\mathbb{R}^2$ sharing a common fixed point p which is a LAS fixed point for both of them, and such that p is repeller for the composition map $f_{2,1}$.

Main results: Parrondo's Paradox-III

ARBITRARY DIMENSION m

Corollary

The following statements hold.

(a) *For all $m \geq 1$ there exist $k \geq 3$ polynomial maps $f_i : \mathbb{R}^m \to \mathbb{R}^m$, for $i \in \{1, \ldots, k\}$, sharing a common fixed point p which is LAS for each map, and such that p is repeller for the composition map $f_{k,k-1,\ldots,1}$. Furthermore, for one-dimensional maps ($m = 1$), this result is optimal on k.*

(b) *For all $m = 2r \geq 2$ there exist 2 polynomial maps $f_1, f_2 :\subseteq \mathbb{R}^{2r} \to \mathbb{R}^{2r}$, sharing a common fixed point p which is LAS for both maps, and such that p is repeller for the composition map $f_{2,1}$.*

A different Parrondo's dynamic paradox

In

- J.S. Cánovas, A. Linero, D. Peralta-Salas. *Dynamic Parrondo's paradox.* Physica D 218 (2006) 177–184.

the authors study global questions combining periodically some simple one-dimensional maps, showing that their combination may produce chaos.

Outline of the talk

One dimensional example

The 3 maps:

$$f_1(x) = -x + 3x^2 - 9x^3 + 164x^5,$$
$$f_2(x) = -x + 5x^2 - 25x^3 + 1259x^5,$$
$$f_3(x) = -x + 2x^2 - 4x^3 + 33x^5.$$

have been chosen in such a way that all them have the origin as a "very weak" LAS fixed point.
Then, it can be proved that the origin of

$$f_{3,2,1}(x) = -x + 90x^4 - 48x^5 + O(6),$$

is repeller, by computing $f^2_{3,2,1} = f_{3,2,1} \circ f_{3,2,1}$.
We have introduced what we will call *stability constants* to systematically study the stability of non-hyperbolic fixed points.

One dimensional example, another surprise

In our example the origin is:

- A LAS fixed point for each $f_i, i = 1, 2, 3$,
- A repeller fixed point for $f_{3,2,1}, f_{2,1,3}$ and $f_{1,3,2}$, BUT
- A LAS fixed point for $f_{3,1,2}, f_{1,2,3}$ and $f_{2,3,1}$.

Two dimensional example

We consider the maps

$$f_1(x,y) = \left(-y + 2x^2 + 6xy, x - 3x^2 + 2xy + 3y^2\right),$$

$$f_2(x,y) = \left(\frac{x}{2} - \frac{\sqrt{3}}{2}y - x\,(x^2+y^2), \frac{\sqrt{3}}{2}x + \frac{1}{2}y - y\,(x^2+y^2)\right).$$

We prove that the origin is a LAS fixed point for both maps f_1 and f_2 and the origin is a repeller fixed point for the composition map $f_{2,1}$.

Recall our local first result in dimension 1

DIMENSION 1

Theorem

The following statements hold:

(a) *Consider two analytic maps f_1, f_2, having a common fixed point p which is LAS. Then, the point p is either LAS or semi-AS for the composition map $f_{2,1}$ and both possibilities may happen.*

(b) *There are $k \geq 3$ polynomial maps f_i, $i = 1, 2, \ldots, k$ sharing a common fixed point p which is LAS for all them and such that p is a repeller fixed point for the composition map $f_{k,k-1,\ldots,1}$.*

Tools for its proof:

- Stability in the orientation preserving case

$$f(x) = x + a_m x^m + O(m+1), \text{ with } a_m \neq 0,\ m \geq 2.$$

- Stability in the orientation reversing case: the stability constants. Similar quantities are introduced in
  - F.M. Dannan, S. Elaydi, V. Ponomarenko. *Stability of hyperbolic and nonhyperbolic fixed points of one-dimensional maps*. J. Difference Equations and Appl. 9 (2003), 449–457.
- The normal forms for non-hyperbolic fixed points of Chen (1968) and Takens (1973), when f is orientation preserving (resp. reversing):
  (a) $g(x) = \lambda x$, with $|\lambda| \neq 1$ and $\lambda > 0$ (resp. $\lambda < 0$),
  (b) $g(x) = x + (\pm x)^{m+1} + cx^{2m+1}$ (resp. $g(x) = -x \pm x^{m+1} + cx^{2m+1}$),

The stability constants-I

Given a $\mathcal{C}^\omega(\mathcal{U})$ function of the form

$$f(x) = -x + \sum_{j\geq 2} a_j x^j,$$

$$f^2(x) := f\circ f(x) = x + \sum_{j\geq 3} W_j(a_2,\ldots,a_j)x^j.$$

If f is not an involution (i.e. $f^2\neq \mathrm{Id}$), we define the *stability constant of order* $\ell\geq 3$, as V_ℓ, where

$$V_3 := W_3(a_2,a_3) \quad \text{and} \quad V_\ell := W_\ell(a_2,\ldots,a_\ell) \text{ if } W_j=0,\ j=3,\ldots,\ell-1.$$

Theorem

Let f be an analytic map in $\mathcal{U}\subseteq\mathbb{R}$ such that $f(0)=0$, $f'(0)=-1$. If f is not an involution, then there exists $m\geq 1$ such that $V_3=V_4=V_5=\cdots=V_{2m}=0$ and $V_{2m+1}\neq 0$. Moreover, if $V_{2m+1}<0$ (resp. $V_{2m+1}>0$), the origin is LAS (resp. repeller).

The stability constants-II

Theorem

Let f be an analytic map in $\mathcal{U} \subseteq \mathbb{R}$ such that $f(0) = 0$, $f'(0) = -1$. If f is not an involution, then there exists $m \geq 1$ such that $V_3 = V_4 = V_5 = \cdots = V_{2m} = 0$ and $V_{2m+1} \neq 0$. Moreover, if $V_{2m+1} < 0$ (resp. $V_{2m+1} > 0$), the origin is LAS (resp. repeller). Moreover:

$$\begin{aligned} V_3 =& -2{a_2}^2 - 2a_3, \\ V_5 =& -6\,a_4a_2 + 4\,{a_3}^2 - 2\,a_5, \\ V_7 =& 3\,a_2a_3a_4 - 8\,a_6a_2 + 13\,a_3a_5 - 4\,{a_4}^2 - 2\,a_7, \\ V_9 =& \frac{242}{17}\,a_2a_3a_6 - \frac{121}{17}\,a_2a_4a_5 - 10\,a_8a_2 \\ &+ \frac{358}{17}\,a_3a_7 - 10\,a_4a_6 + \frac{69}{17}\,{a_5}^2 - 2\,a_9. \end{aligned}$$

Recall our local first result in dimension 2:

DIMENSION 2

Theorem

There exist polynomial maps f_1 and f_2 in $\mathbb{R}^2$ sharing a common fixed point p which is a LAS fixed point for both of them, and such that p is repeller for the composition map $f_{2,1}$.

Tools for the proof:

- The Birkhoff normal form

$$f_B(z,\bar{z}) = \lambda z\Big(1+\sum_{j=1}^{m} B_j (z\bar{z})^j\Big) + O(2m+2),$$

where $z = x + yi$ and B_j are complex numbers, for non $(2m+1)$-resonant elliptic fixed points.

- The *Birkhoff stability constants*, $V_j = \mathrm{Re}(B_j)$.

The Birkhoff stability constants-I

Lemma

For $m \in \mathbb{N}$, consider a $\mathcal{C}^{2m+2}$-map f with an elliptic fixed point $p \in \mathcal{U}$, non $(2m+1)$-resonant. Let B_m be its first non-vanishing Birkhoff constant. If $V_m = \mathrm{Re}(B_m) < 0$ (resp. $V_m = \mathrm{Re}(B_m) > 0$), then the point p is LAS (resp. repeller).

The Birkhoff stability constants-II

Set

$$g(z,\bar{z}) = \lambda z + \sum_{m+j=2}^{3} a_{m,j} z^m \bar{z}^j + O(4),$$

where $z \in \mathbb{C}$, $\lambda \in \mathbb{C}$, $|\lambda| = 1$. Then the first Birkhoff constant is

$$B_1 = B_1(g) = \frac{P(g)}{\lambda^2 (\lambda - 1)(\lambda^2 + \lambda + 1)}, \tag{1}$$

where

$$\begin{aligned} P(g) = & \left(|a_{11}|^2 + a_{21}\right) \lambda^4 - a_{11}\left(2a_{20} - \overline{a}_{11}\right) \lambda^3 \\ & + \left(2|a_{02}|^2 - a_{11}a_{20} + |a_{11}|^2\right) \lambda^2 \\ & - \left(a_{11}a_{20} + a_{21}\right) \lambda + a_{11}a_{20}, \end{aligned}$$

Outline of the talk

Recall the first global question

If f_1 has "complicated" dynamics, can we choose some f_2, with "simple" dynamics, and such that the composition map $f_{2,1}$ has simple dynamics?

Global case 1: Lyness recurrences

Given the Lyness k-periodic recurrence:

$$x_{n+2} = \frac{a_n + x_{n+1}}{x_n},$$

where $\{a_n\}$ is a cycle of k positive numbers, i.e. $a_{n+k} = a_n$, it holds that:

$$(x_1, x_2) \xrightarrow{f_{a_1}} (x_2, x_3) \xrightarrow{f_{a_2}} (x_3, x_4) \xrightarrow{f_{a_3}} (x_4, x_5) \xrightarrow{f_{a_4}} (x_5, x_6) \xrightarrow{f_{a_5}} \cdots$$

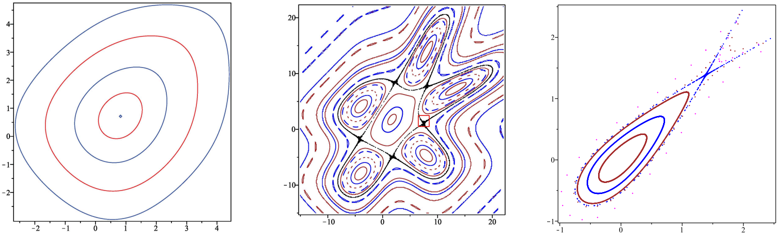
and in general,

$$f_{[k]}(x_1, x_2) = (x_{k+1}, x_{k+2}).$$

where

$$f_{[k]} := f_{a_k,\ldots,a_2,a_1} = f_{a_k} \circ \cdots \circ f_{a_2} \circ f_{a_1}.$$

Lyness recurrences: main results



$k \in \{1, 2, 3, 6\}$ $\qquad$ $k = 4$ $\qquad$ $k = 5$

Figure 1: Different possible behaviors of the orbits of $h_{[k]}$, according to k. Here $h_{[k]}$ is conjugated to $f_{[k]}$.

Lyness recurrences: main results

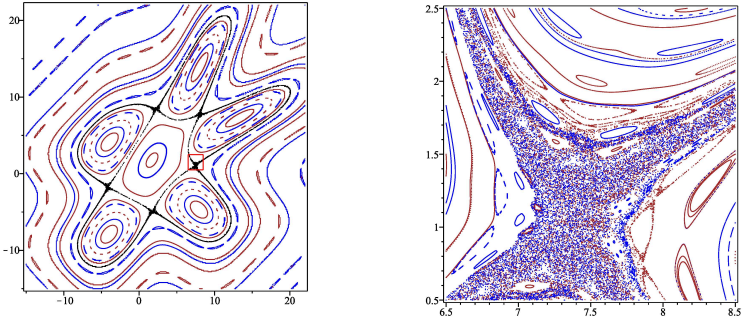


Figure 2: Some orbits of $h_{0.001,7,4,2}$ and a zoom with much more orbits.

Lyness recurrences: main results

$\circ$	$f_{[2]}$	$f_{[4]}$	$f_{[5]}$
$f_{[1]}$	$f_{[3]}$	$f_{[5]}$	$f_{[6]}$
$f_{[2]}$	$f_{[4]}$	$f_{[6]}$	—

Table: Some composition maps.

$\circ$	integrable	chaotic	non integrable
integrable	integrable	non integrable	integrable
integrable	chaotic	integrable	——

Table: Regularization of chaos.

Lyness recurrences: Tools-1

Proposition. Let $f : \mathcal{U} \subset \mathbb{R}^2 \to \mathbb{R}^2$ be a $\mathcal{C}^2(\mathcal{U})$ map such that $f(0,0) = (0,0) \in \mathcal{U}$ and $Df(0,0)$ is diagonalizable, with real eigenvalues λ and μ, $\lambda\mu \neq 0$. If f has a meromorphic first integral H in $\mathcal{U}$ then there exists $(0,0) \neq (p,q) \in \mathbb{Z}^2$ such that $\lambda^p \mu^q = 1$.

The above result can be used to prove that for most values of a_j the map $f_{[5]}$ is NOT MEROMORPHICALLY INTEGRABLE.

Lyness recurrences: Tools-2

Theorem. (i) The non-autonomous k-periodic Lyness recurrences have invariants of the form

$$V(x,y,n)=\frac{\Phi_n(x,y)}{xy},\quad \Phi_n(x,y)\quad \text{polynomials of degree 4},$$

if and only if $k\in\{1,2,3,6\}$.
(ii) The first integrals of the associated maps $f_{[k]}$, for $k\in\{1,2,3,6\}$, corresponding to the invariants found in (i) are:

$$V_a(x,y)=\frac{a+(a+1)x+(a+1)y+x^2+y^2+x^2y+xy^2}{xy},$$

$$V_{b,a}(x,y)=\frac{ab+(a+b^2)x+(b+a^2)y+bx^2+ay^2+ax^2y+bxy^2}{xy},$$

$$V_{c,b,a}(x,y)=\frac{ac+(a+bc)x+(c+ab)y+bx^2+by^2+cx^2y+axy^2}{xy},$$

$$V_{f,e,d,c,b,a}(x,y)=\frac{af+(a+bf)x+(f+ae)y+bx^2+ey^2+cx^2y+dxy^2}{xy}.$$

Outline of the talk

Recall the second global question

If each f_i is conjugated with some g_i, $i = 1, 2$, is there some relation between the dynamics of $f_{2,1}$ and the dynamics of $g_{2,1}$?

Global case 2: McMillan–Gumovski–Mira recurrences-I

Recall the recurrences considered by McMillan–Gumovski–Mira (which correspond to a special subfamily of QRT maps):

$$x_{n+2} = -x_n + \frac{\alpha x_{n+1}}{1 + x_{n+1}^2} \quad \text{with} \quad \alpha > 0.$$

Performing the change of variables $x_n = \sqrt{\alpha}\, y_n$, they write as

$$y_{n+2} = -y_n + \frac{y_{n+1}}{\beta + y_{n+1}^2} \quad \text{with} \quad \beta = 1/\alpha > 0.$$

So both recurrences are equivalent.

Global case 2: McMillan–Gumovski–Mira recurrences-II

We consider the corresponding 2-periodic recurrences

$$x_{n+2} = -x_n + \frac{\alpha_n x_{n+1}}{1 + x_{n+1}^2},$$

and

$$x_{n+2} = -x_n + \frac{x_{n+1}}{\beta_n + x_{n+1}^2},$$

where $\{\alpha_n\}_n$ is a 2-periodic cycle of positive values and $\beta_n = 1/\alpha_n, n \geq 0$.

Global case 2: McMillan–Gumovski–Mira recurrences-III

To study them we use once more the composition maps:

$$f_{\alpha_2,\alpha_1} := f_{\alpha_2} \circ f_{\alpha_1} \quad \text{and} \quad g_{\beta_2,\beta_1} := g_{\beta_2} \circ g_{\beta_1},$$

where

$$f_\alpha(x,y) = \left(y, -x + \frac{\alpha y}{1+y^2}\right)$$

and

$$g_\beta(x,y) = \left(y, -x + \frac{y}{\beta + y^2}\right).$$

Global case 2: McMillan–Gumovski–Mira recurrences-IV

We show:

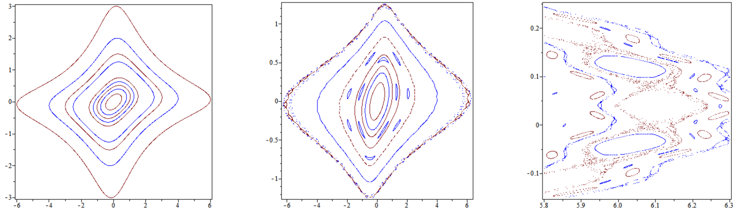
- The map f_{α_2,α_1} numerically exhibits all the features of a non-integrable perturbed twist map.

BUT

- The map $g_{1/\alpha_2,1/\alpha_1}$ has the first integral

$$V(x,y) = \frac{1}{\alpha_1}x^2 + \frac{1}{\alpha_2}y^2 + x^2y^2 - xy.$$

Global case 2: McMillan–Gumovski–Mira recurrences-V



(a) (b) (c)

Figure 1: In (a) and (b), some orbits of the maps $g_{1/b,1/a}$ and $f_{b,a}$ with $a = 2$, $b = 1/2$ are depicted. In (c), a detail of (b) is shown.

Moltes gràcies per la vostra atenció