Internal dynamics in wandering domains

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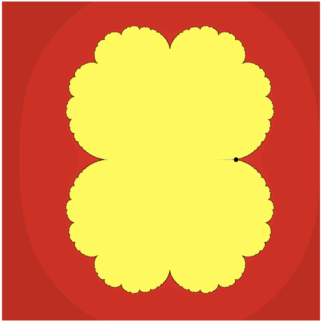
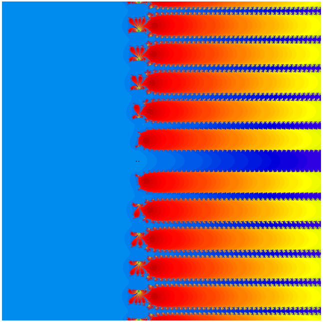
Holomorphic dynamics in $\mathbb{C}$

The complex plane decomposes into two **totally invariant sets**:

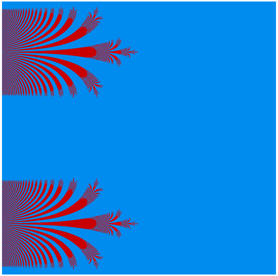
- **The Fatou set (or stable set)**: basins of attraction of attracting or parabolic cycles, Siegel discs (irrational rotation domains), ... **[Fatou classification Theorem, 1920]**
- **The Julia set (or chaotic set)**: the closure of the set of repelling periodic points (boundary between the different stable regions).



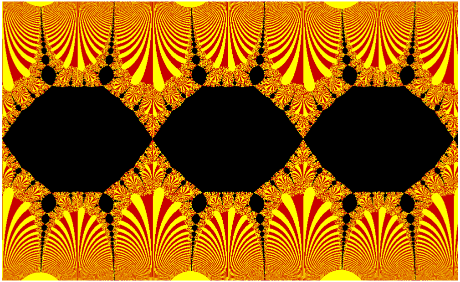


Transcendental dynamics

- If $f : \mathbb{C} \to \mathbb{C}$ has an essential singularity at infinity we say that f is **transcendental**.
- Transcendental maps may have Fatou components that are not basins of attraction nor rotation domains:
  - U is a **Baker domain** of period 1 if $f^n |_U \to \infty$ loc. unif.
  - U is a **wandering domain** if $f^n(U) \cap f^m(U) = \emptyset$ for all $n \neq m$.



$z + 1 + e^{-z}$



$z + 2\pi + \sin(z)$

Wandering domains: a program

Quite uncharted territory . . .

- They do not exist for rational maps [Sullivan'82] – only for transcendental.
- "Recently" discovered – First example (an infinite product) due to Baker in the 80's (multiply connected, escaping to infinity)
- It is not easy to construct examples – WD are not associated to periodic orbits.
- They do not exist for maps with a finite number of **singular values**.

Singular values

Holomorphic maps are local homeomorphisms everywhere except at the **critical points**

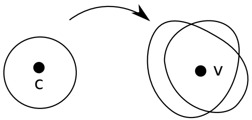
$$Crit(f) = \{c \mid f'(c) = 0\}.$$

Singular values:

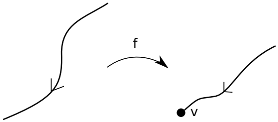
$$S(f) = \{v \in \mathbb{C} \mid \text{not all branches of } f^{-1} \text{ are well defined in a nbd of } v\}.$$

These can be

- **Critical values** $CV = \{v = f(c) | c \in Crit(f)\}$;
- **Asymptotic values** $AV = \{a = \lim_{t\to\infty} f(\gamma(t)); \gamma(t) \to \infty\}$, or
- accumulations of those.



critical value



asymptotic value

Special classes

Some classes of maps are singled out depending on their singular values.

- The **Speisser class or finite type maps**:

$$\mathcal{S} = \{f \text{ ETF (or MTF) such that } S(f) \text{ is finite}\}$$

Example: $z \mapsto \lambda \sin(z)$

Maps in S have **NO WANDERING DOMAINS.**

[Eremenko-Lyubich'87, Goldberg+Keen'89]

- The **Eremenko-Lyubich class**

$$\mathcal{B} = \{f \text{ ETF (or MTF) such that } S(f) \text{ is bounded}\}$$

Example: $z \mapsto \lambda \frac{z}{\sin(z)}$.

Maps in $\mathcal{B}$ have **NO ESCAPING WANDERING DOMAINS**.

[Eremenko-Lyubich'87]

Types of wandering domains

- $\{f^n\}$ form an equicontinuous family on a Wandering domain U.
- All limit functions are constant in $J(f) \cap \overline{P(f)}$ [Baker'02].

$$L(U) = \{a \in \mathbb{C} \cup \infty \mid \exists n_k \to \infty \text{ with } f^{n_k} \to a\}$$

Types of wandering domains:

$$\text{U} \quad \text{is} \begin{cases} \textbf{escaping} & \text{if } L(U) = \{\infty\} \\ \textbf{oscillating} & \text{if } \{\infty, a\} \subset L(U) \text{ for some } a \in \mathbb{C}. \\ \textbf{``bounded''} & \text{if } \infty \notin L(U). \end{cases}$$

▸ PICTURE

Open question: Do "bounded" domains exist at all?

Oscillating WD in class $\mathcal{B}$

$\rightarrow$ a recent result [Bishop'15, F-Jarque-Lazebnik'18, Martí-P-Shishikura'18]

Examples of wandering domains

Examples of wandering domains are not abundant. Usual methods are:

- **Lifiting of maps** of $\mathbb{C}^*$ [Herman'89, Henriksen-F'09]. The relation with the singularities is limited to the finite type possibilities.
- Infinite products and clever modifications of known functions [Bergweiler'95, Rippon-Stallard'08'09...]
- **Approximation theory** [Eremenko-Lyubich'87]. No control on the dynamics of the global map (singular values, etc).

State of the art

Postsingular set: $P(f)$ = forward iterates of $S(f)$.

- Examples of WD exist: simply and multiply connected, fast escaping and slowly escaping, bounded (as sets) and unbounded, oscillating, univalent, ...
  [Baker, Rippon+Stallard, Eremenko+Lyubich, F+Henriksen, Sixsmith, ...]

- The relation between limit functions and the singular values is partially understood ($L(U) \in P(f)'$).
  [Baker, Bergweiler *et al*]

- The relation between simply connected WD and $P(f)$ is partially understood. [Rempe-Gillen + Mihailevic-Brandt'16, Baranski+F+Jarque+Karpinska'18]

- **Internal dynamics???**

Lifting of holomorphic maps of $\mathbb{C}^*$: An example

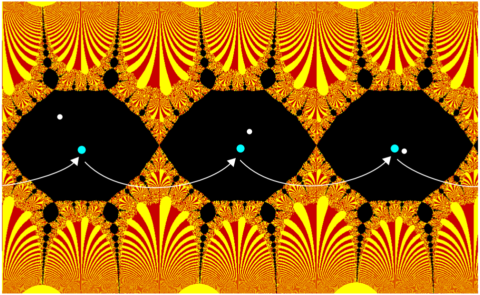
$F(w) = w\exp(\frac{1}{2}(z - \frac{1}{z}))$ and $f(z) = z + 2\pi + \sin(z)$ are semiconjugate via $w = e^{iz}$.

$$\begin{array}{ccc} \mathbb{C} & \xrightarrow{z+2\pi+\sin(z)} & \mathbb{C} \\ e^z \big\downarrow & & \big\downarrow e^z \\ \mathbb{C}^* & \xrightarrow{w\exp(\frac{1}{2}(z-\frac{1}{z}))} & \mathbb{C}^* \end{array}$$

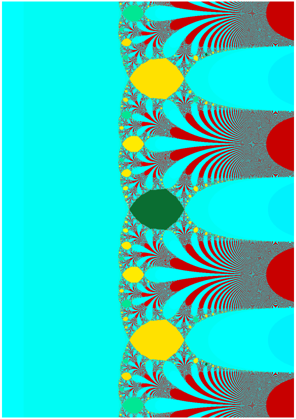
- F has a superattracting basin around $z = 0$ which lifts to a **wandering domain**.

BUT ORBITS REMEMBER WHERE THEY CAME FROM!!!

Lifting of holomorphic maps of $\mathbb{C}^*$: Examples
Lifts of superattracting basins

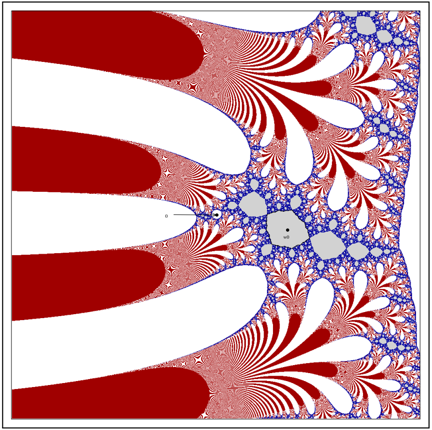


$z + 2\pi + \sin(z)$
WD (black).



$\ln \lambda_1 + 2z - e^z$
Wandering D. (yellow).

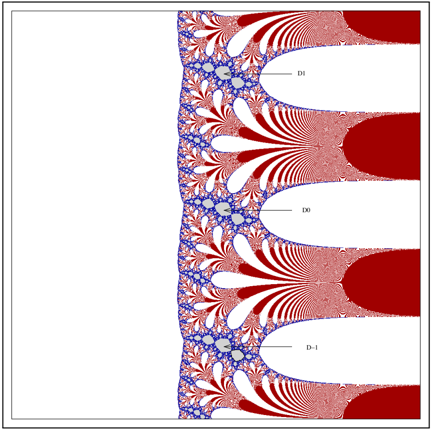
Lift of a Siegel disk



$\lambda_0 w^2 e^{-w}$

Siegel disk (gray).

Basin of 0 (white).



$\ln \lambda_0 + 2z - e^z$

Wandering domain (gray).

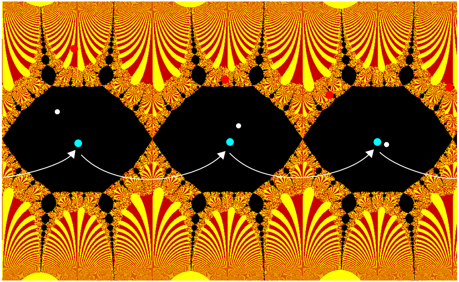
Baker domains (white).

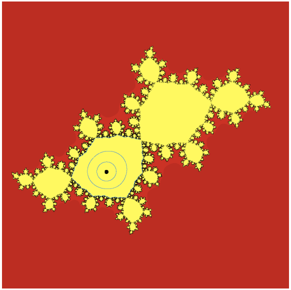
Lifting of holomorphic maps of $\mathbb{C}^*$: Orbits remember

U wandering domain obtained by lifting $V = \exp(U)$

$U_n := f^n(U)$

- V **attracting basin** of a fixed point $p \longrightarrow$ orbits converge to the orbit of $\ln p$, well inside U_n.
- V **parabolic basin** of a fixed point $p \in \partial V \longrightarrow$ orbits converge to the orbit of $\ln p \in \partial U_n$.
- V **Siegel disk** $\longrightarrow$ orbits rotate on the lifts of "invariant curves".





Questions

Hence internal dynamics on WD can be of different types.

Questions

- How special are these examples?
- Can other internal dynamics occur?
- Is there a **"Classification Theorem"** as for periodic components?

A priori there is no reason to believe that because

$$f : U_n \to U_{n+1}$$

is somehow different for each n.
(Non-autonomous dynamics? Forward iterated functions systems?)

BUT, dynamics on **multiply connected** wandering domains are quite well understood [Baker'84, Zheng'06, Rippon-Stallard, Bergweielr-Rippon-Stallard'13]

Internal dynamics

Two prespectives:

- **Orbits move with the wandering domains** (like passengers in a cruise ship follow the ship's trajectory)
- On the other hand there are **intrinsic dynamics relative to each other**, or relative to the domains boundary (like passengers gathering at the buffet for dinner, or going to the ship edges to watch the water).

Internal dynamics: the hyperbolic distance

Intrinsic tool which does not depend on the embedding of the WD in the plane.

- $U_n := f^n(U)$ hyperbolic ($\#\partial U \geq 2$), simply connected.
- $\text{dist}_U(z, w)$ hyperbolic distance between $z, w \in U$.

Schwarz-Pick Lemma

U, V hyperbolic, $f : U \to V$ holomorphic . Then, for all $z, w \in U$,

$$\text{dist}_V(f(z), f(w)) \leq \text{dist}_U(z, w),$$

and "=" occurs iff f is an isometry (univalent case).

Hence $f : U_n \to U_{n+1}$ contracts for all n and

$$\lim_{n\to\infty} \text{dist}_{U_n}(f^n(z), f^n(w)) = c(z, w) \geq 0 \text{ as } n \to \infty$$

Different limits for different pairs of z, w???

First classification theorem

Let U be a simply connected, bounded, wandering domain for an entire map f and let $U_n := f^n(U)$. Define the countable set of pairs

$$E = \{(z, w) \in U \times U \mid f^k(z) = f^k(w) \text{ for some } k \in \mathbb{N}\}.$$

Then, exactly one of the following holds as $n \to \infty$, **for all** $(z, w) \notin E$:

(1) U is **(hyperbolically) contracting**, i.e.

$$\text{dist}_{U_n}(f^n(z), f^n(w)) \longrightarrow c(z, w) \equiv 0;$$

(2) U is **(hyperbolically) semi-contracting**, i.e.

$$\text{dist}_{U_n}(f^n(z), f^n(w)) \longrightarrow c(z, w) > 0;$$

(3) U is **(hyperbolically) eventually isometric**, i.e.

$$\exists N > 0 \text{ such that } \forall n \geq N, \ \text{dist}_{U_n}(f^n(z), f^n(w)) = c(z, w) > 0.$$

First classification theorem: Observations

- Lifts of **Siegel disks** are **eventually isometric** .
- Lifts of **attracting basings** AND **parabolic basins** are **contracting** .

We can actually refine the contracting condition to distinguish between these two cases.

Definition

U is **strongly contracting** if $\exists c \in (0,1)$ such that for all $z, w \in U$

$$\text{dist}_{U_n}(f^n(z), f^n(w)) = \mathcal{O}(c^n).$$

Even stronger, U is **super-contracting** if for all $z, w \in U$

$$\lim_{n\to\infty} \left(\text{dist}_{U_n}(f^n(z), f^n(w))\right)^{1/n} = 0.$$

First classification theorem: Observations

- Lifts of **attracting basins** are **strongly contracting**
- Lifts of **parabolic basins** are **never strongly contracting** because

$$\frac{k}{n} \leq \mathrm{dist}_{U_n}(f^n(z), f^n(w)) \leq \frac{K}{n}, \quad k, K \in \mathbb{R}.$$

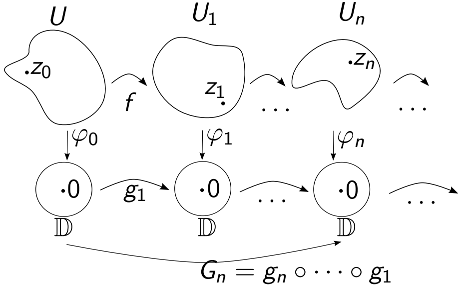
- **Super-contracting** WD are e.g. those containing orbits with infinitely many critical points.

The **Hyperbolic distortion** helps us differentiate the different cases.

MORE

A tool: non-autonomus discrete systems

We choose a base point $z_0 \in U$, $z_n := f^n(z_0)$ and choose Riemann maps $\varphi_n : U_n \to \mathbb{D}$ such that $\varphi_n(z_n) = 0$.



The maps $g_n : \mathbb{D} \to \mathbb{D}$ (and hence G_n) are **Inner Functions**.

A tool: non-autonomus discrete systems

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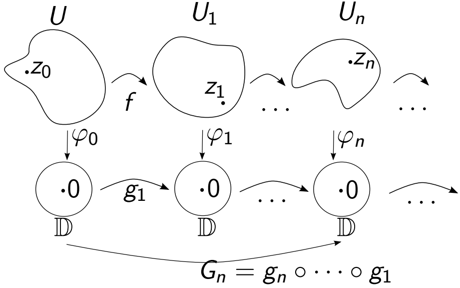


The maps $g_n : \mathbb{D} \to \mathbb{D}$ (and hence G_n) are **Inner Functions**.

We have a **non-autonomous dynamical system of self-maps of $\mathbb{D}$**.

The rates of contraction depend on the values of $g_n'(0)$
(in fact $\sum_n (1 - |g_n'(0)|)$).

Orbit interactions with the boundary

Convergence of orbits to the boundary is a delicate concept, because the shape of U_n may degenerate.
We use the following definition.

Definition (Convergence to the boundary)
We say that the orbit of $z \in U$ **converges to the boundary** (of U_n) if and only if

$$\lim_{n\to\infty} \text{dist}(f^n(z), \partial U_n) = 0.$$

Other definitions are possible, (e.g. taking into account the largest disk contained in U_n). In any case, the following result holds.

Second classification theorem

Let U be a simply connected, wandering domain for f entire and let $U_n := f^n(U)$. Then, exactly one of the following holds.

(1) **(CONV)** For all $z \in U$

$$\lim_{n\to\infty} \operatorname{dist}(f^n(z), \partial U_n) = 0$$

that is, **all orbits converge to the boundary**;

(2) **(BDAWAY)** For all points $z \in U$ and every $n_k \to \infty$

$$\lim_{k\to\infty} \operatorname{dist}(f^{n_k}(z), \partial U_{n_k}) \neq 0,$$

that is, **all orbits stay away from the boundary**; or

(3) **(BUNGEE)** Neither (1) nor (2), i.e. **all orbits oscillate.**

Convergence to the boundary: Observations

- If U is the lift of a **parabolic basin**, then U is of type (1) **(CONV)**.
- If U is the lift of a Siegel disk, or an **attracting basin**, then U is of type (2) **(BDAWAY)**.
- No **(BUNGEE)** WD can come from a lift.

Question

In case (1) **(CONV)**, does there exist a **distinguished point** in the boundary which attracts all orbits? (Denjoy-Wolf for this setting?)

Realization

The classification theorems leave us with a 3×3 table of possibilities.

	$\to \partial$	$\not\to \partial$	oscillating
contracting	Lift of parab. b.	Lift of attrac. b.	?
semi-contracting	?	?	?
ev. isometric	?	Lift of Siegel Disk	?

Question: Can all cases be realized?

ANSWER: YES.

Realization Theorem

Theorem

There exist transcendental entire functions f_i, $i = 1, 2, 3$, having a sequence of bounded, simply connected, escaping wandering domains realizing the following conditions.

(a) Every orbit under f_1 converges to the boundary;

(b) Every orbit under f_2 stays away from the boundary;

(c) Every orbit under f_3 comes arbitrarily close to the boundary but does not converge to it.

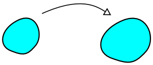
Moreover, each of the examples f_i, $i = 1, 2, 3$, can be chosen to be (hyperbolically) attracting, semi-attracting or eventually isometric.

Proof: By approximation theorem (not explicit examples) :-(

Gràcies per la vostra atenció!!

Escaping WD

Escaping WD



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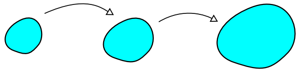

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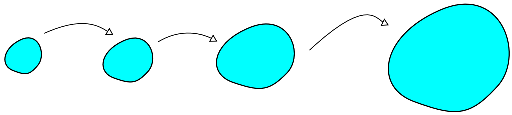

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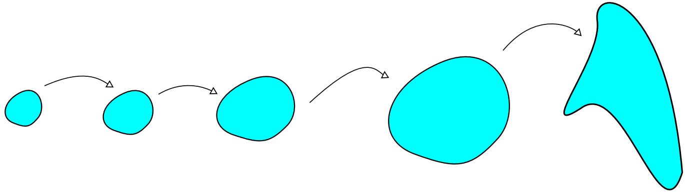

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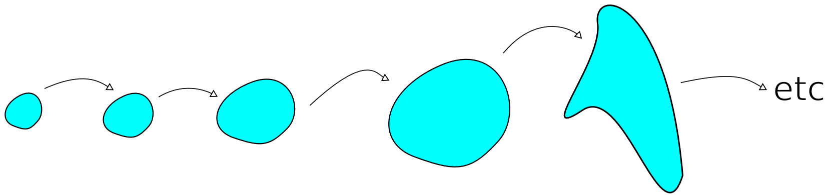

Oscillating WD

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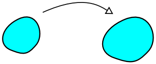

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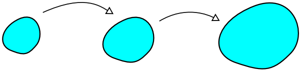

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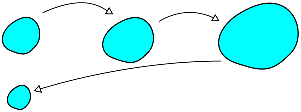

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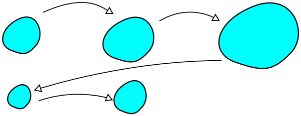

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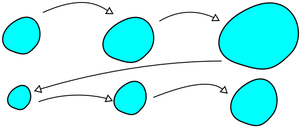

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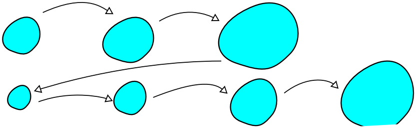

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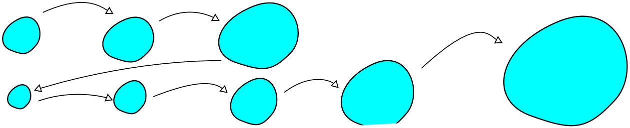

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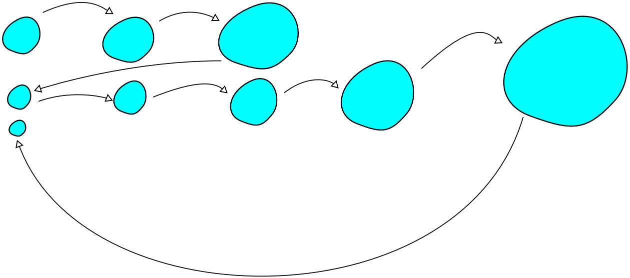

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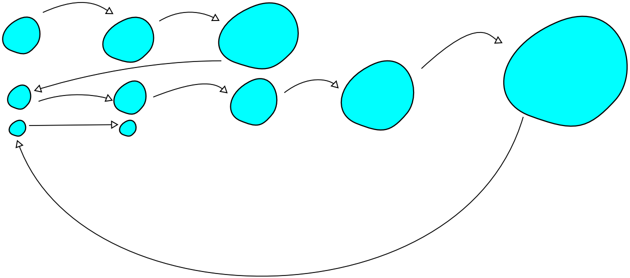

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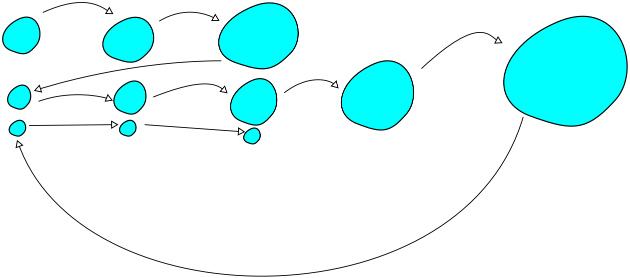

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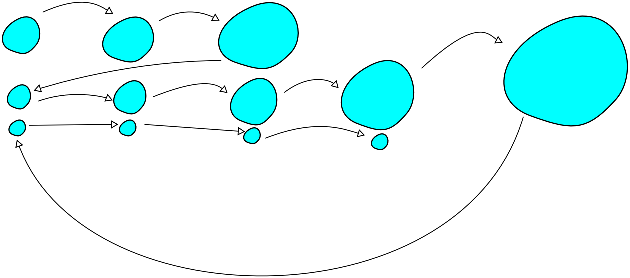

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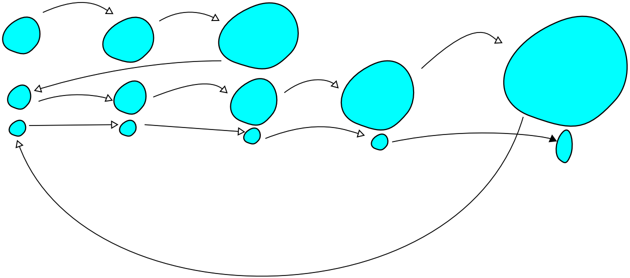

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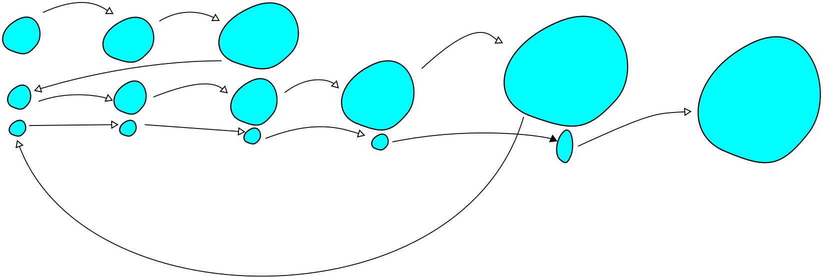

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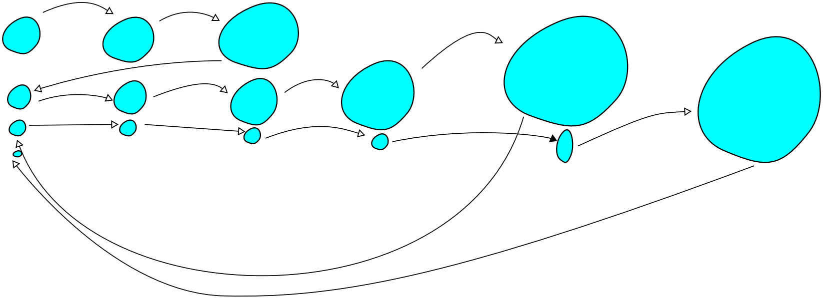
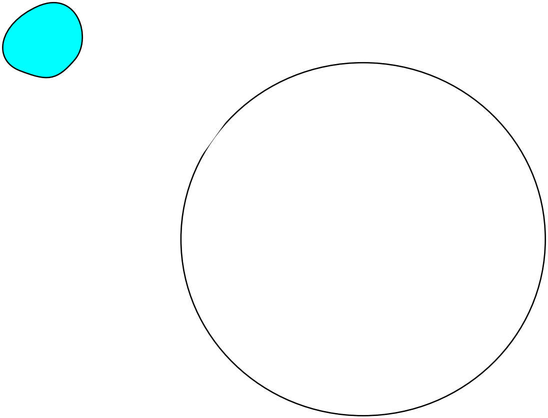


"Bounded" WD



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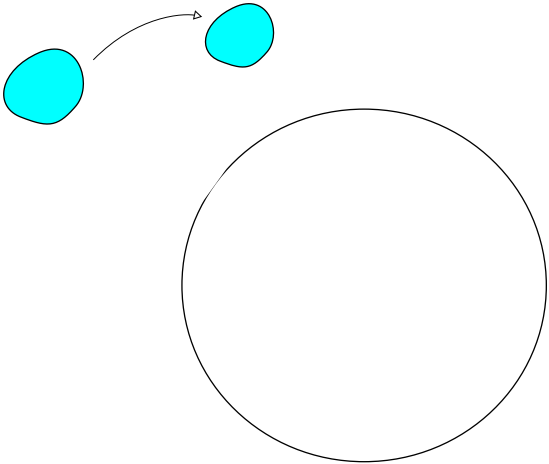
"Bounded" WD



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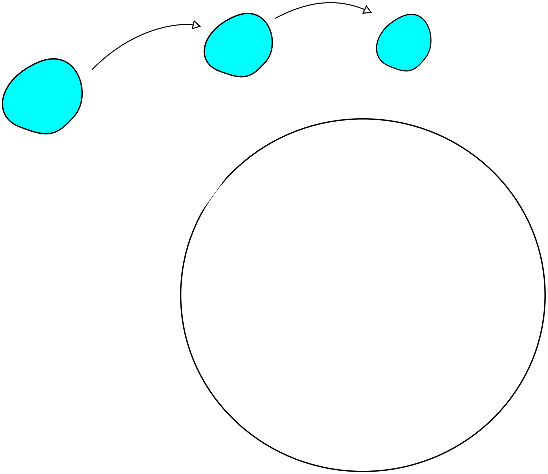
"Bounded" WD



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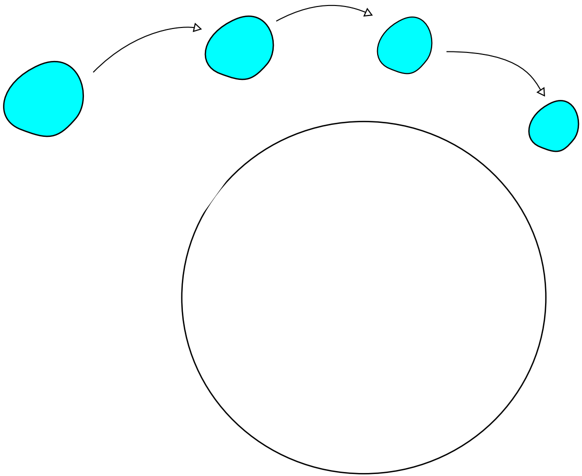

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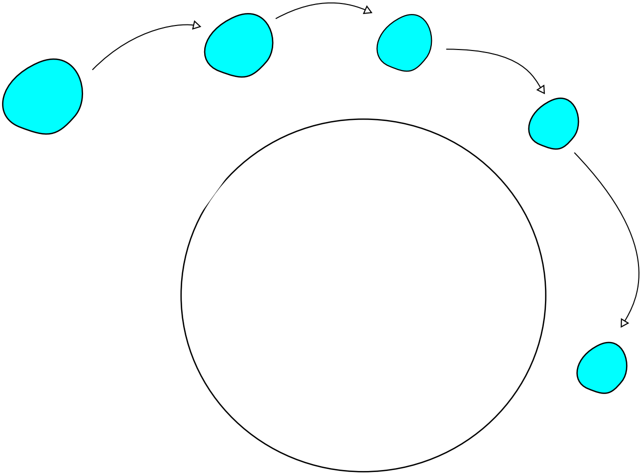

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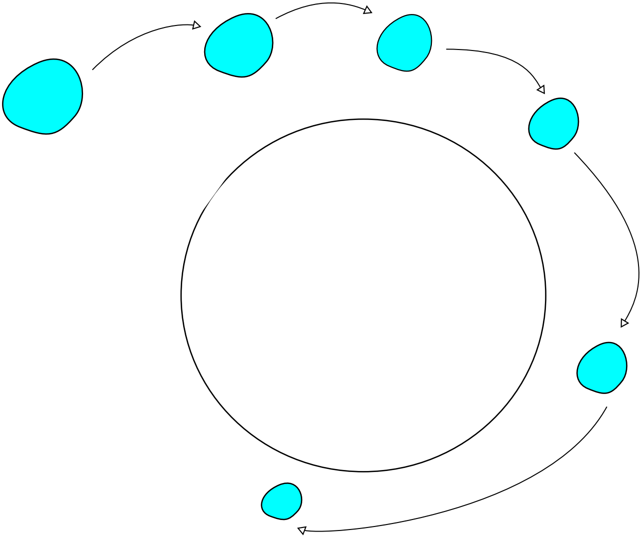

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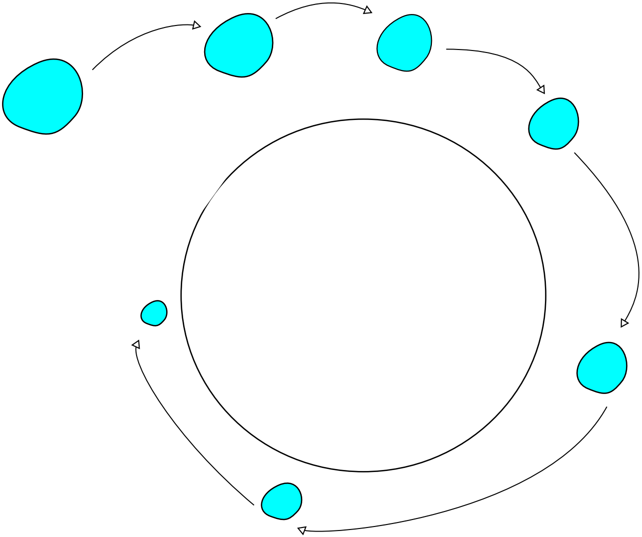

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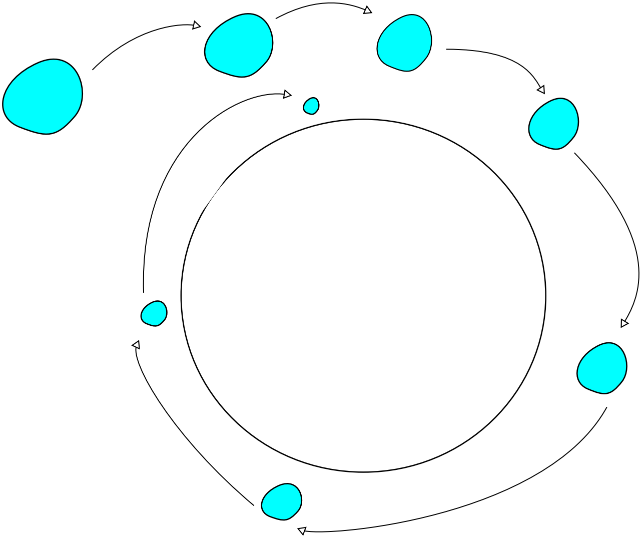

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