Invariant manifolds in Celestial Mechanics

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Outline

The comprehension of simple models allows both to understand the mechanisms that explain natural phenomena and to design spacecraft mission.

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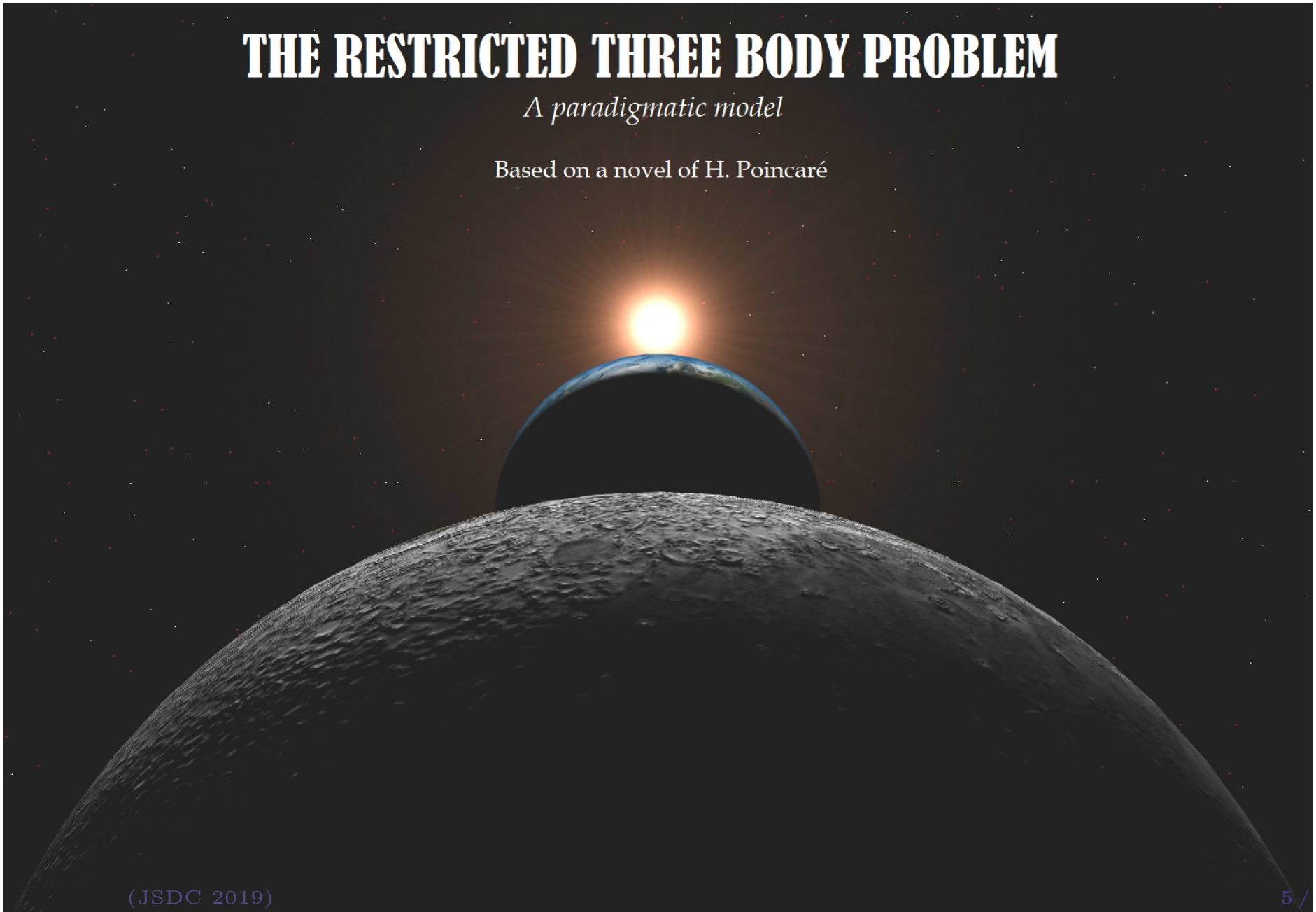
All models are wrong, but some are useful.

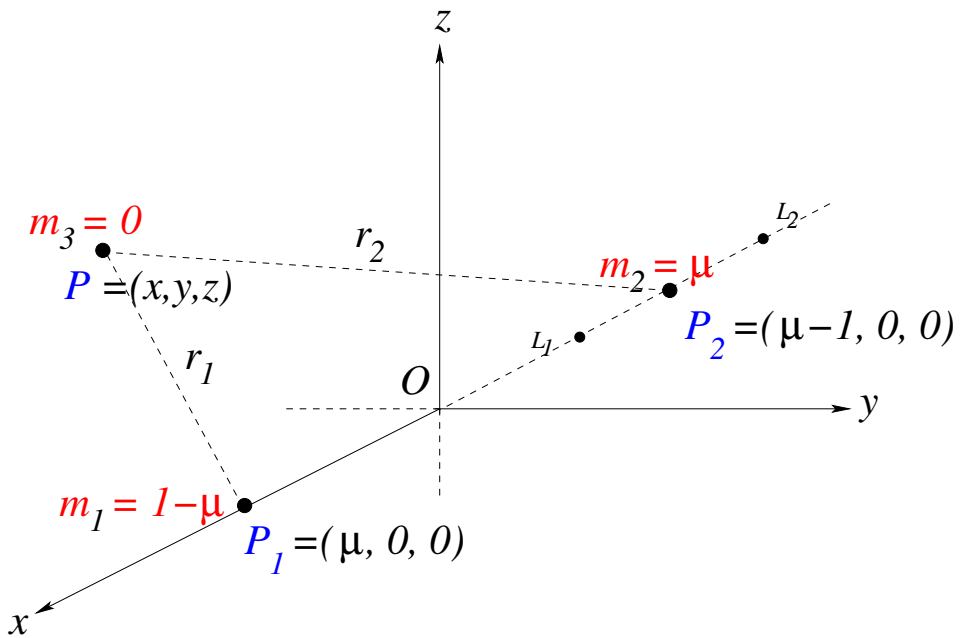
George E. P. Box (1918–2013)

Outline


THE RESTRICTED THREE BODY PROBLEM
A paradigmatic model
Based on a novel of H. Poincaré

The Circular Restricted Three Body Problem

The motion in a rotating system of a (massless) particle under the gravitational attraction of two main bodies (primaries) moving in a circular orbit



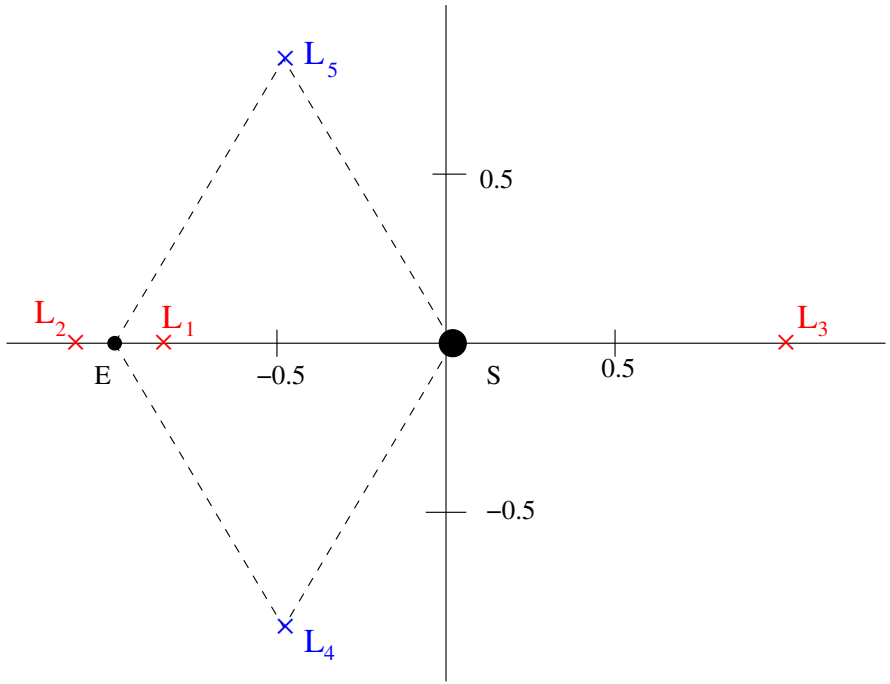
$$H(x,y,z,p_z,p_y,p_z) = \tfrac{1}{2}(p_x^2 + p_y^2 + p_z^2) - xp_y + yp_x - \frac{1-\mu}{r_1} - \frac{\mu}{r_2},$$

$$r_1 = \sqrt{(x-\mu)^2 + y^2 + z^2}, \quad r_2 = \sqrt{(x-\mu+1)^2 + y^2 + z^2}.$$

Planar Circular Restricted Three Body Problem (CRTBP)

Equilibrium points:

- L_1, L_2, L_3 collinear points
- L_4, L_5 triangular points



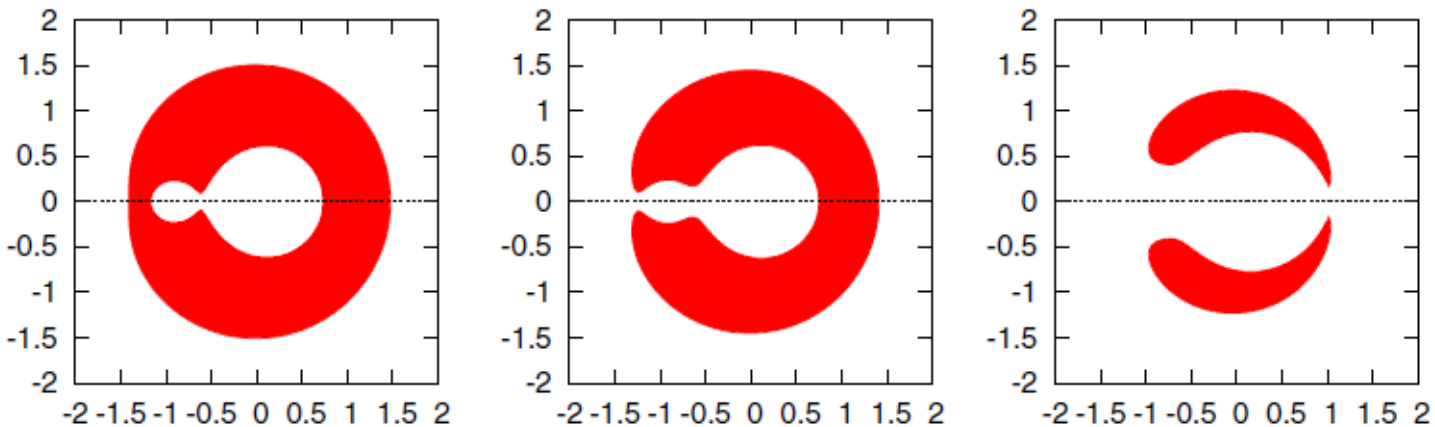
Energy (Jacobi constant):

$$C_J = -2h = x^2 + y^2 + 2\frac{1-\mu}{r_1} + 2\frac{\mu}{r_2} + \mu(1-\mu) - v^2$$

Hill's region

For a fixed value of the energy, the zero velocity curves (surfaces) enclosed the Hill's regions where the motion is allowed.

$$v^2 = x^2 + y^2 + 2\frac{1-\mu}{r_1} + 2\frac{\mu}{r_2} + \mu(1-\mu) - C_J \geq 0$$



Configuration space (x, y)

Dynamics around the equilibrium points

- The triangular points, $L_{4,5}$, are linearly stable for small values of the mass parameter
- The collinear points, L_i, $i = 1, 2, 3$ are unstable:

$$\mathrm{Spec}(D\boldsymbol{f}(L_i)) = \{\lambda, -\lambda, i\omega_v, -i\omega_v, i\omega_p, -i\omega_p\},$$

Linear:	$\pm\lambda$		$\pm i\omega_p$		$\pm i\omega_v$	
	saddle	$\times$	**center**	$\times$	**center**	
			$\uparrow$		$\uparrow$	
Non-linear:	inv. manif.		planar p.o.		vertical p.o.	

$$\underbrace{\qquad\qquad\qquad\qquad}_{\text{Cantor set of 2D tori}}$$

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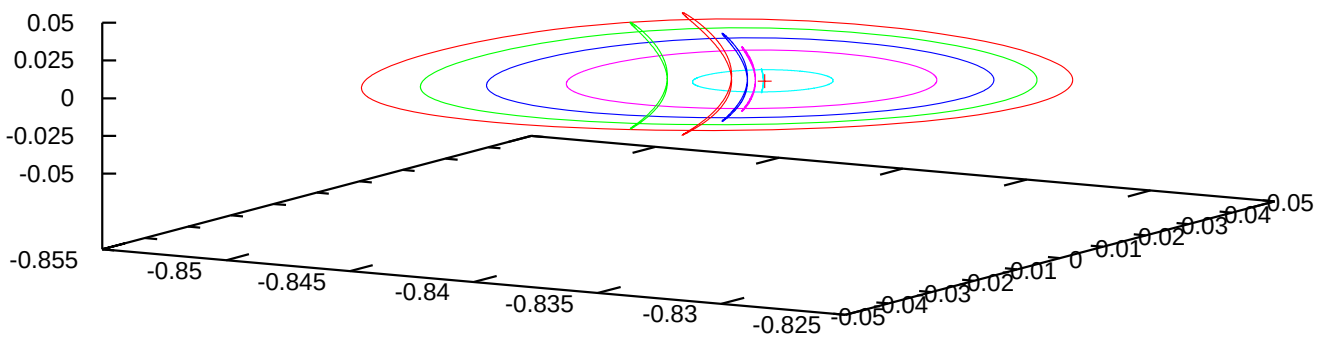
At each energy level $H = h$ $\exists$ one planar and one vertical periodic orbits (Lyapunov p.o.). Along the families, bifurcations appear (halo orbits).



Credits: J.M. Mondelo

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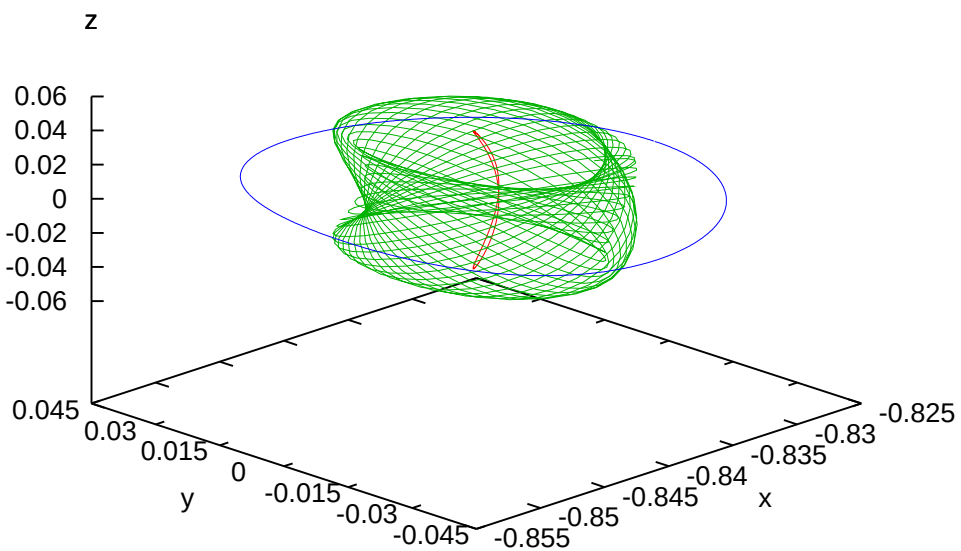
At each energy level $H = h$ $\exists$ a family of quasi-periodic orbits (inv. tori)



Credits: J.M. Mondelo

Invariant manifolds in CRTBP

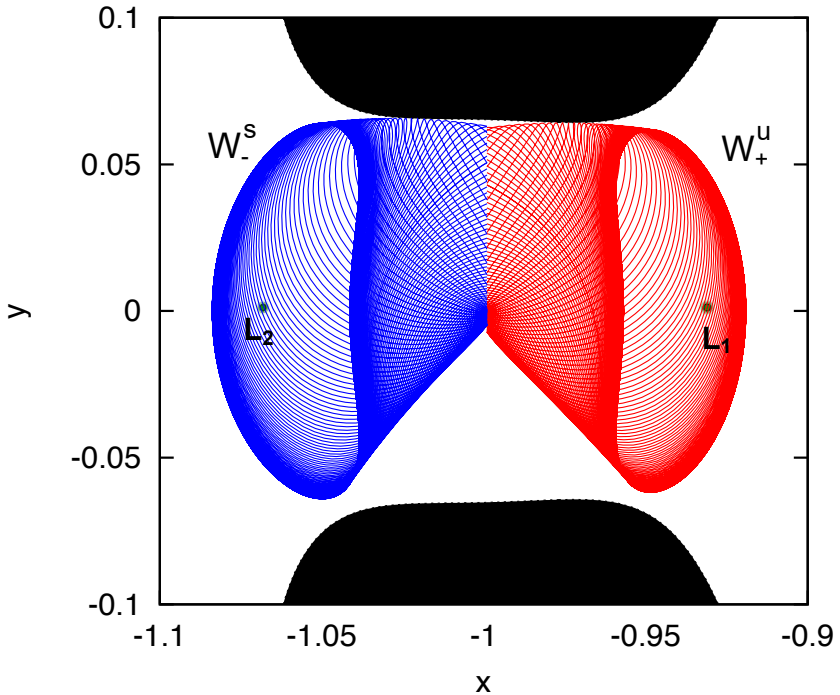
For X an unstable invariant object (equilibrium point, periodic or quasi periodic orbit):

$$W^u(X) = \{\mathbf{q} \in \mathbb{R}^6 \mid \lim_{t \to -\infty} d(\phi_t(\mathbf{q}), X) = 0\},$$
$$W^s(X) = \{\mathbf{q} \in \mathbb{R}^6 \mid \lim_{t \to +\infty} d(\phi_t(\mathbf{q}), X) = 0\},$$

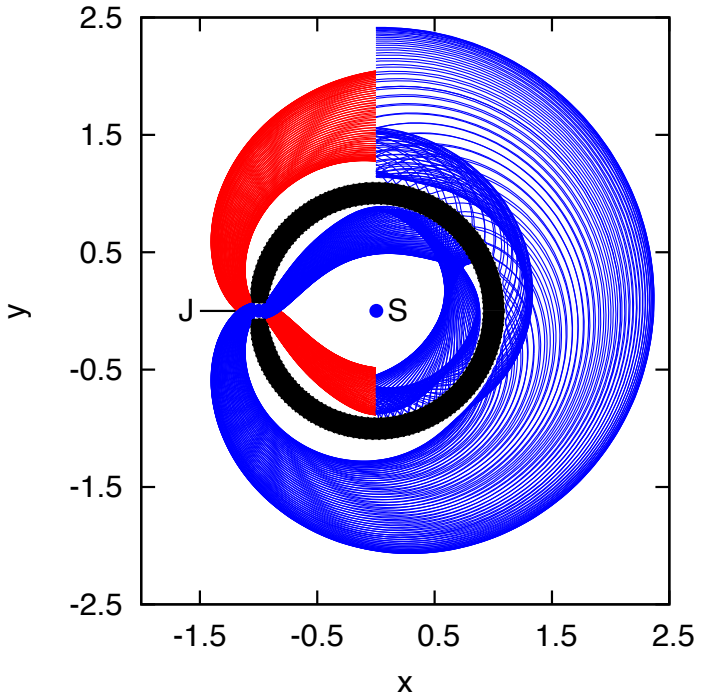
where ϕ_t is the flow and $d(\phi_t(\mathbf{q}), X)$ is the distance between $\phi_t(\mathbf{q})$ and X.

Invariant manifolds in CRTBP

Typical behavior of right $W_+^{u/s}$ and left $W_-^{u/s}$ branches associated to planar Lyapunov orbits around L_1 and L_2



Secondary's region

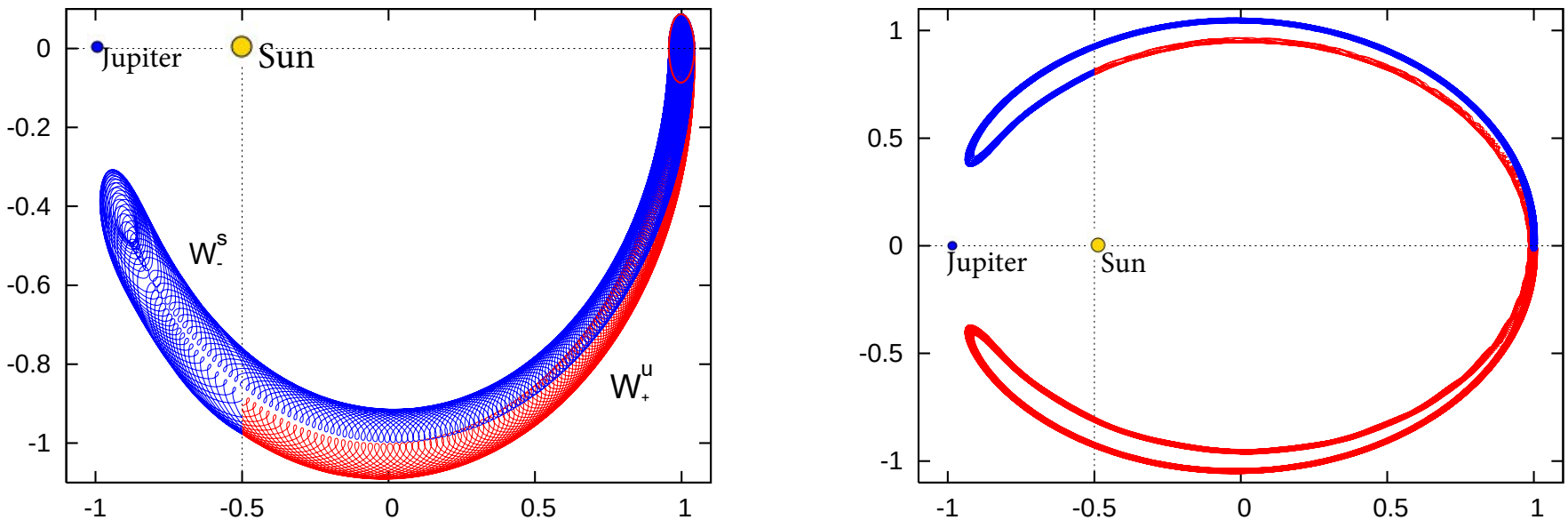


Outer and inner regions

Configuration space (x, y)

Invariant manifolds in CRTBP

Typical behavior of right $W_+^{u/s}$ and left $W_-^{u/s}$ branches associated to planar Lyapunov orbits around L_3



Configuration space (x, y)

Homoclinic and heteroclinic orbits

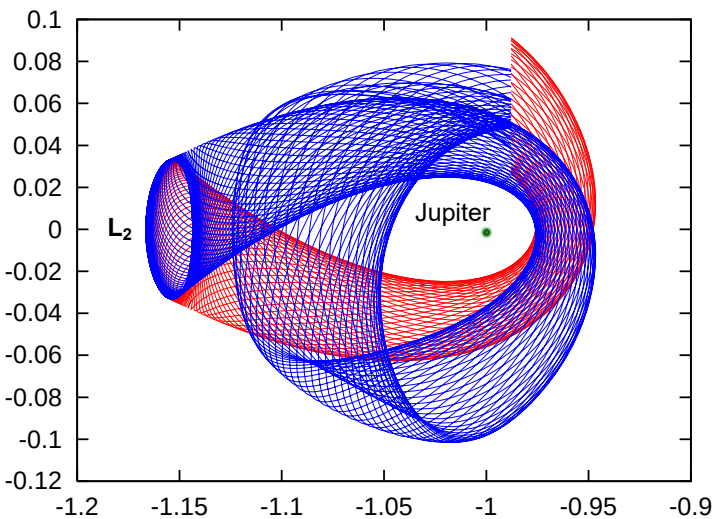
If there exists a transversal intersection of the stable and unstable i.m., the dynamics of the nearby orbits is complex and rich

- $\exists$ a homo/heteroclinic connection: orbit that tends forward ($t \to +\infty$) and backward ($t \to -\infty$) to the hyperbolic object X/Y
- existence of infinitely many periodic orbits close to it (and performing a similar path)
- existence of homoclinic connections of *higher* order

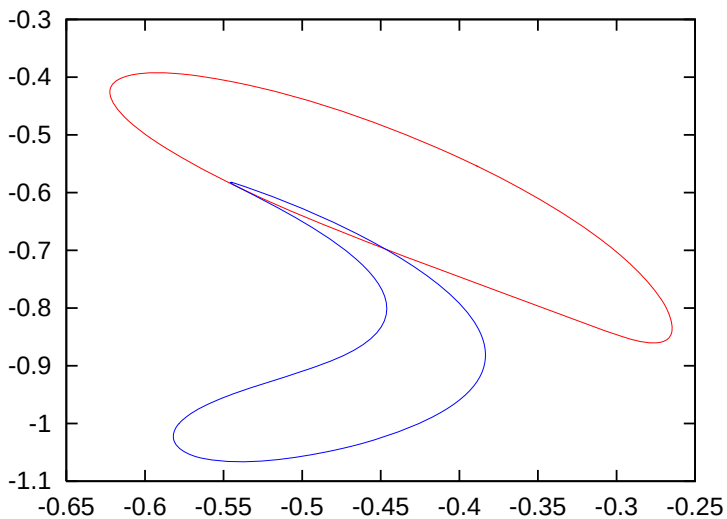
Homoclinics to planar Lyapunov orbits around $L_{1,2}$

For a fixed energy h, a planar Lyapunov orbits OL_i, $i = 1, 2, 3$, and a fixed section Σ

- $W^{u/s}(OL_i)$ are 2-dimensional (tubes)
- $W^{u/s}(X) \cap \Sigma$ are S^1–like closed curves.



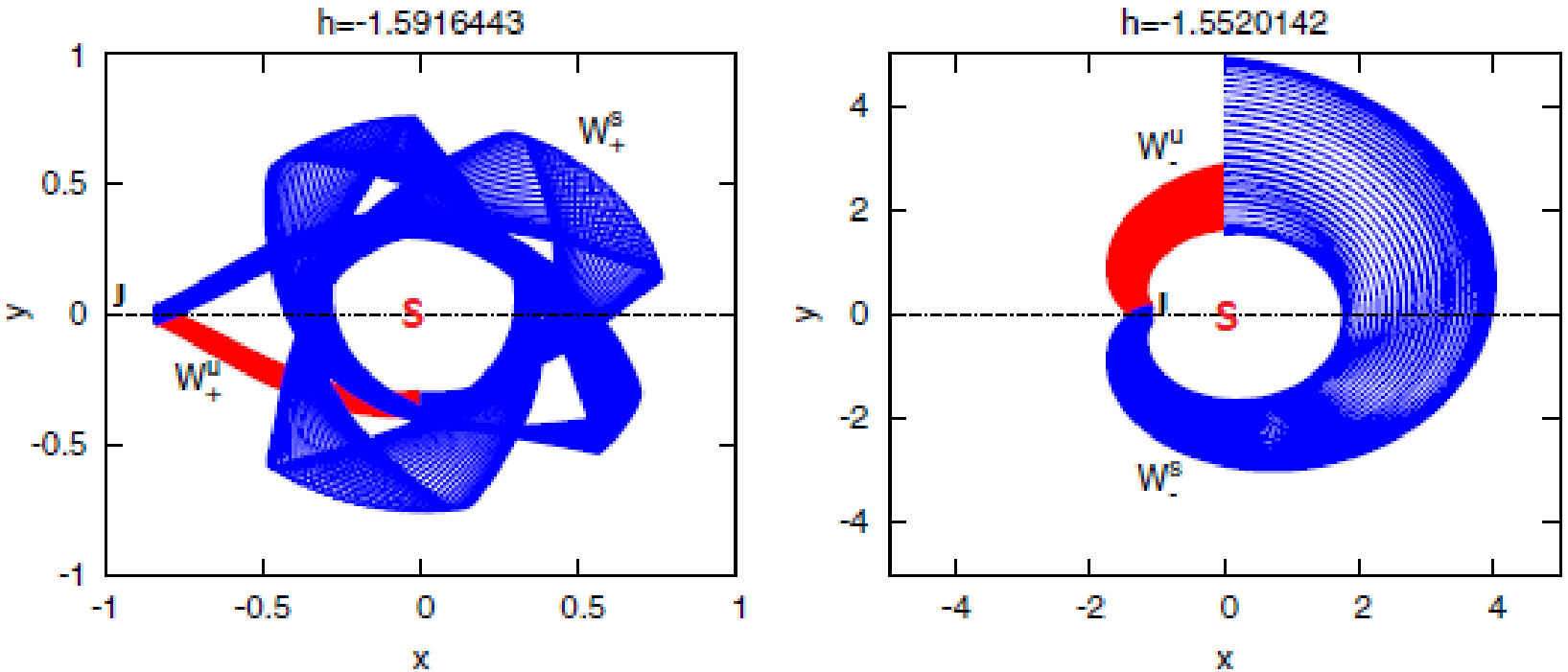
(x, y)-plane



(p_x, p_y)-plane

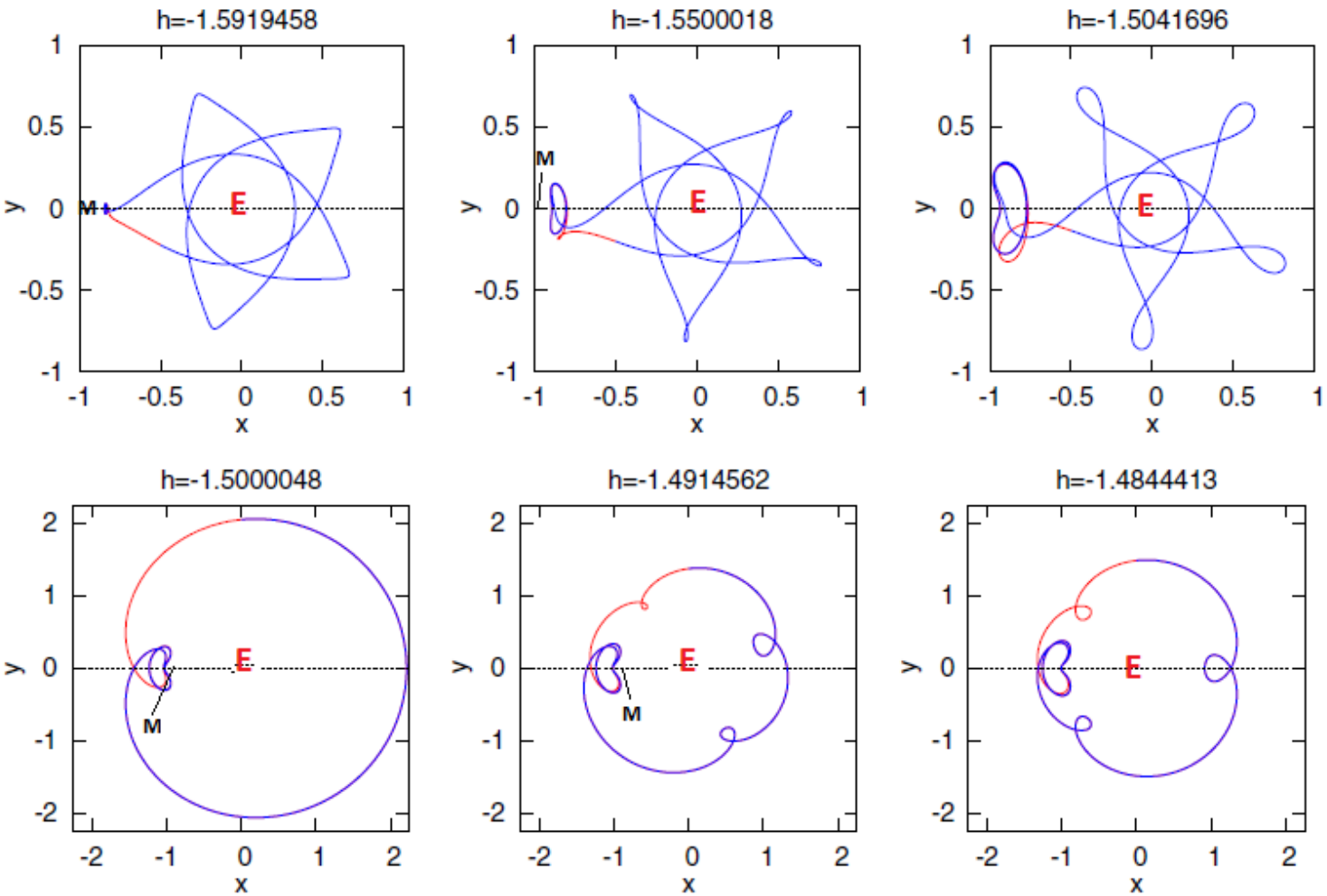
Homoclinics to planar Lyapunov orbits around $L_{1,2}$

Surrounding the big primary: $W_+^{u/s}(OL_1)$ (left) or $W_-^{u/s}(OL_2)$ (right)



Homoclinics to planar Lyapunov orbits around $L_{1,2}$

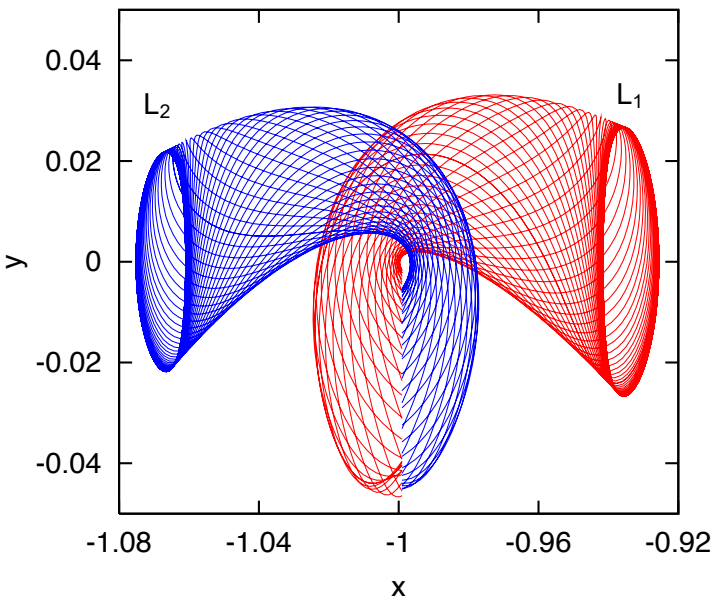
Earth-Moon RTBP - homoclinics surrounding the Earth



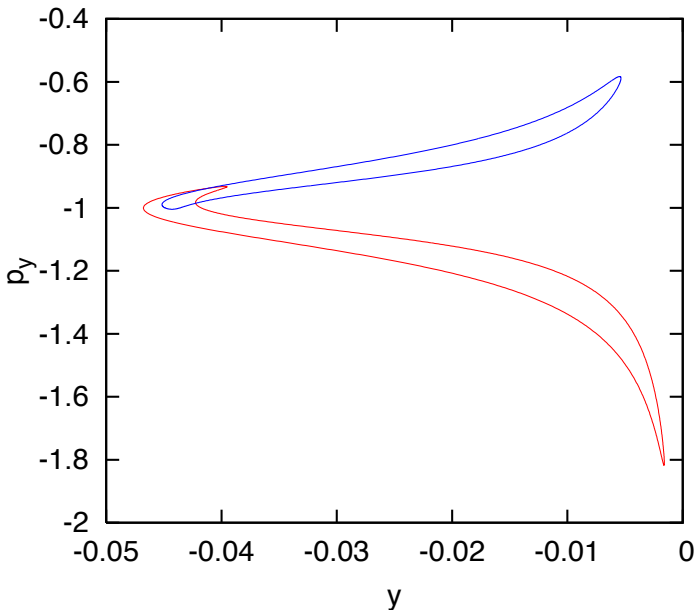
Heteroclinic to Lyapunov orbits around $L_{1,2}$

Heteroclinic connections: $W^u(OL_1) \cap W^s(OL_2)$

Sun - Jupiter CRTBP



(x, y)-plane

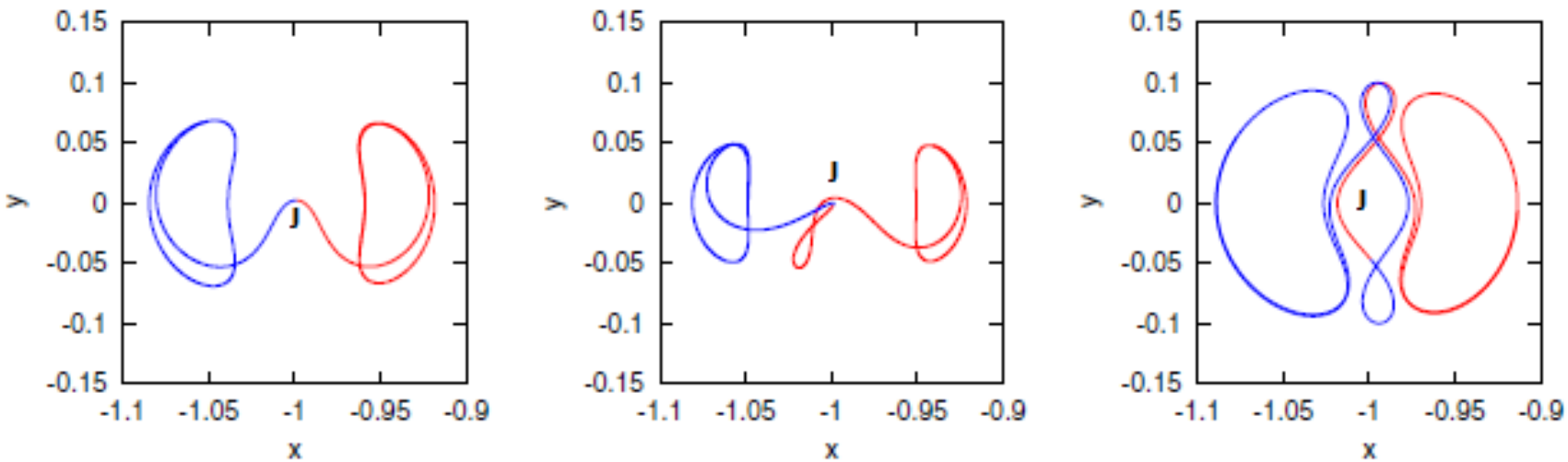


(y, p_y)-plane

Heteroclinic to Lyapunov orbits around $L_{1,2}$

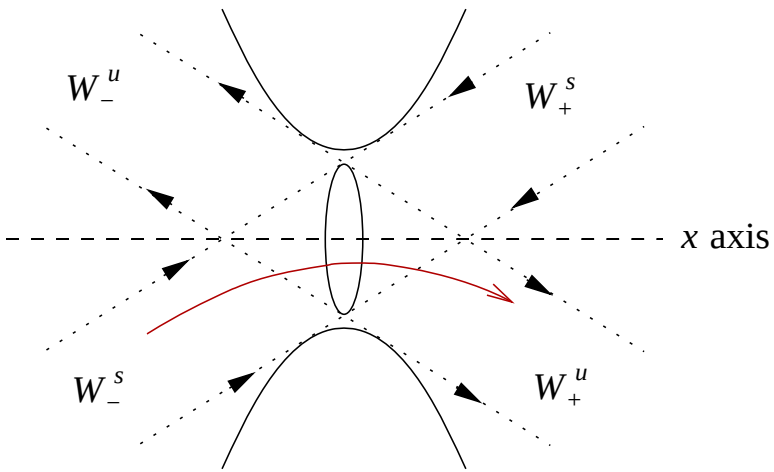
Heteroclinic connections: $W^u(OL_1) \cap W^s(OL_2)$

Sun - Jupiter CRTBP

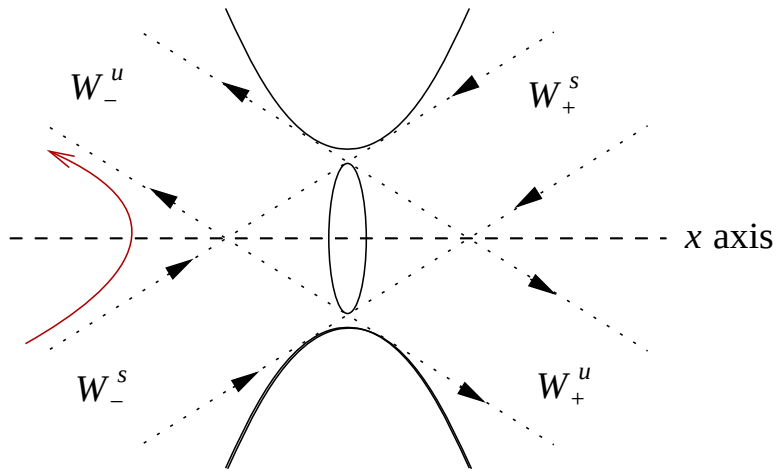


Transit and non-transit orbits in planar CRTBP

A trajectory approaching a p.o., either forward or backward in time from one of the three regions, is considered *transit* if it traverses the bottleneck corresponding to the LPO and goes to the next region



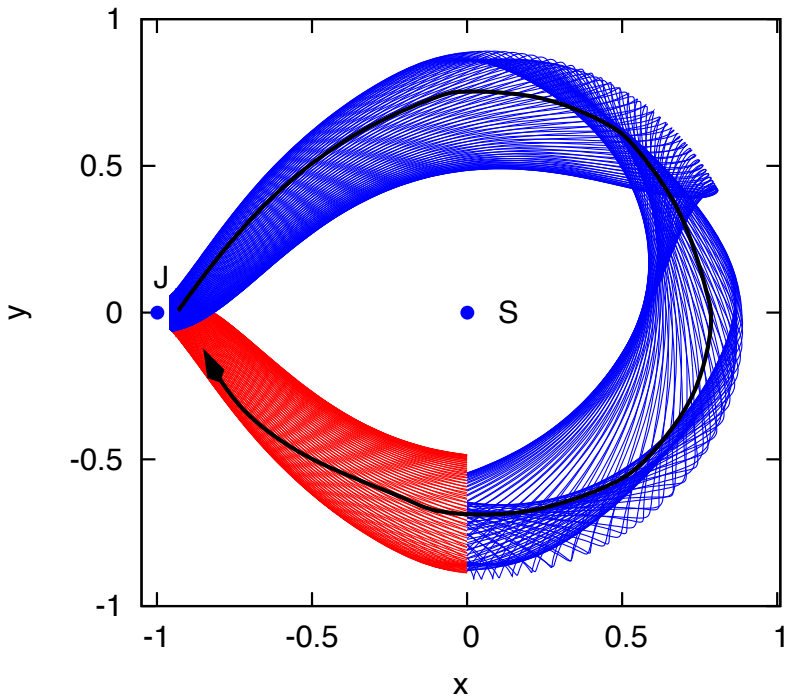
Transit orbit

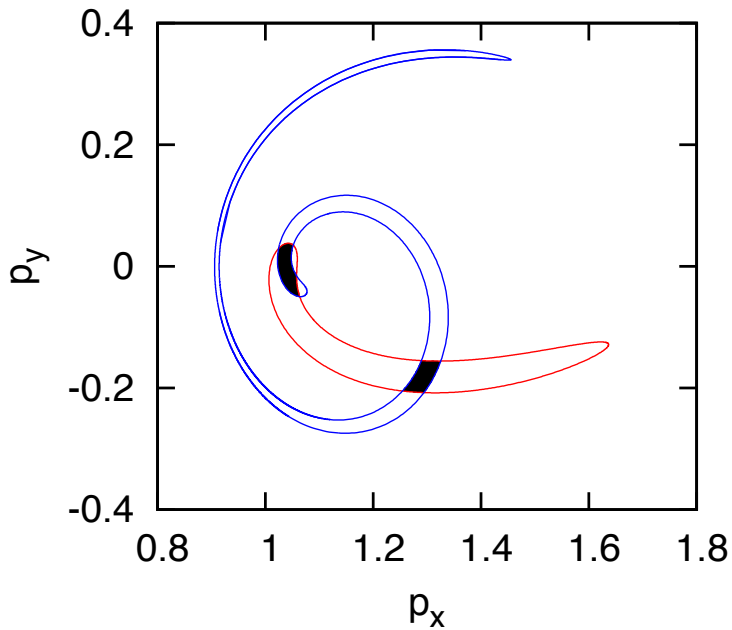


Non-transit orbit

Transit orbits are known to lie in the *interior* of the invariant manifold tubes of the p.o., that separate them from non–transit orbits (Conley,1968; McGehee 1969)

Transit and non-transit orbits in planar CRTBP

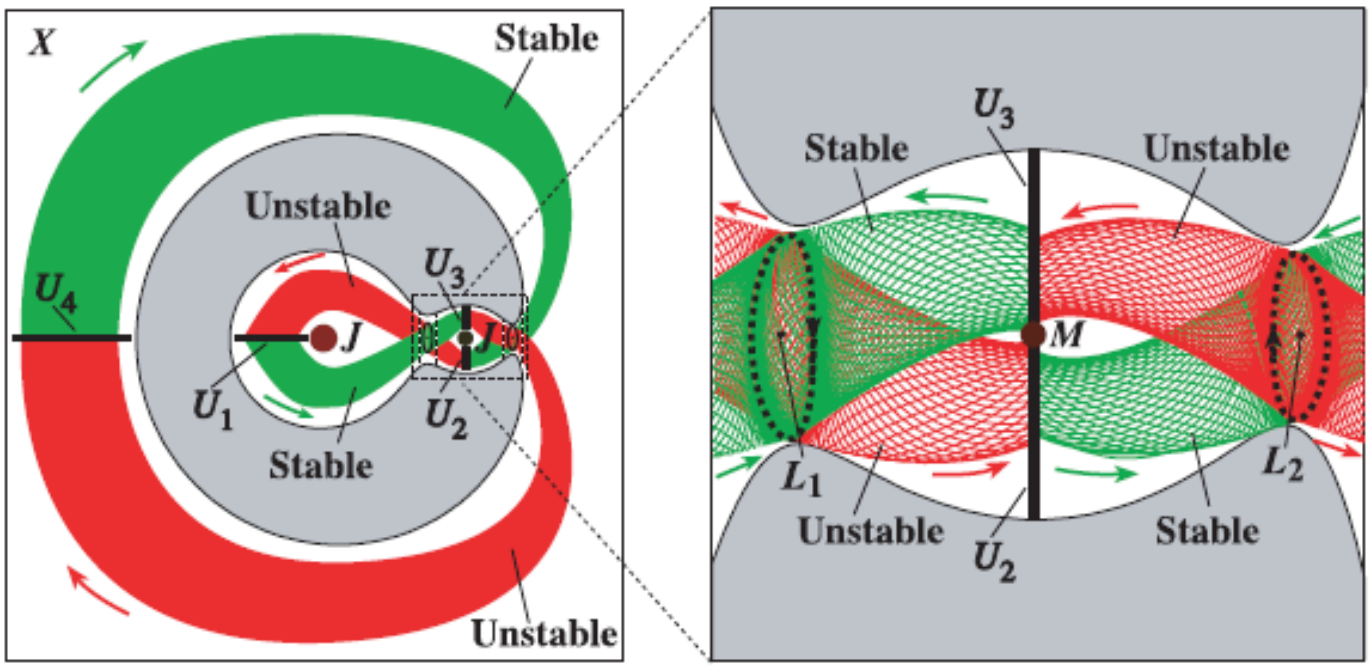




Transit from Jupiter region → inner region → Jupiter region:
the orbits may lie in the interior of both invariant manifolds W^u *and* W^s

Transit and non-transit orbits in planar CRTBP

Transition chain

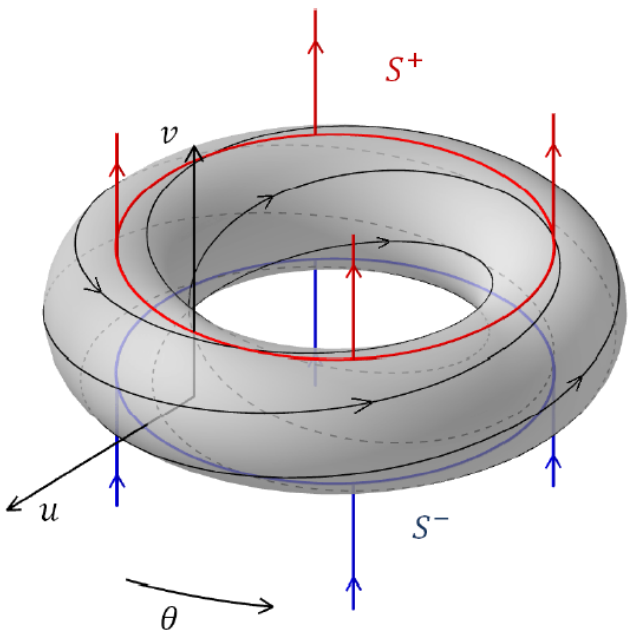


Credits: Gomez et al. Nonlinearity (2004)

Other manifolds

Ejection-Collision Orbit (ECO): the distance between the particle and a primary tends to zero when $t \to t_0^+$ (ejection) and $t \to t_1^-$ (collision).

- Key point: heteroclinic connections between different equilibrium points living on the collision manifold (regularization of the singularity at the primary): a 2-d torus with two circles of equilibrium points



3d invariant manifolds $W^u(S^+)$ ejection orbits , $W^s(S^-)$ collision orbits

Credits: Ollè, Rodríguez, Soler, CNSNS (2017)

Other manifolds

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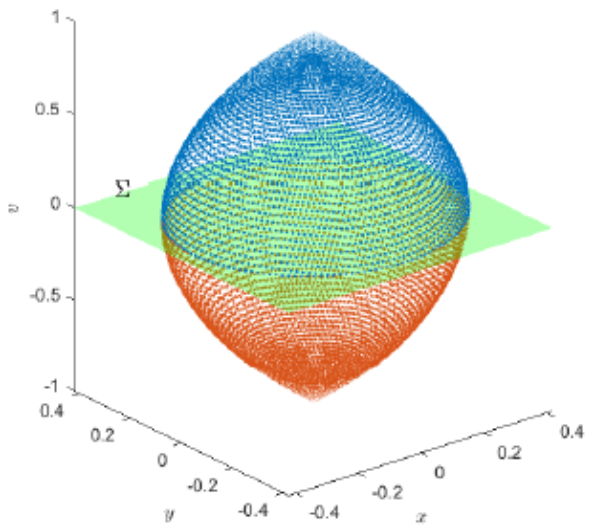
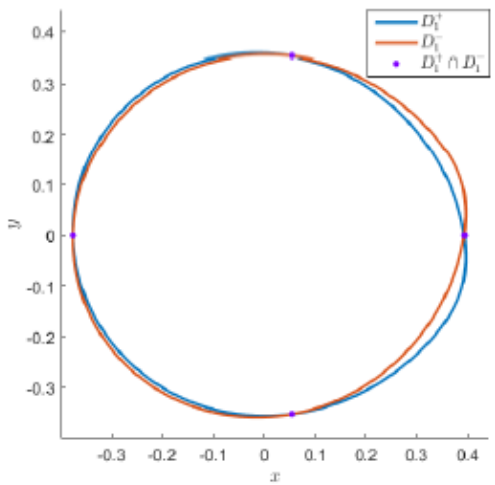
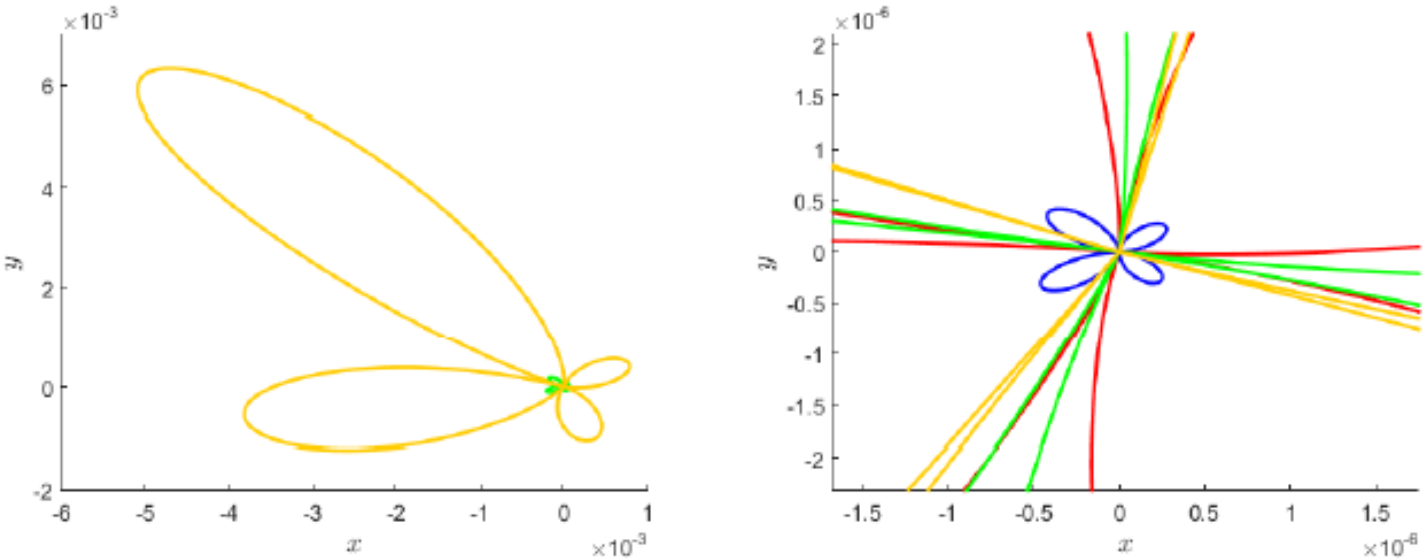




First intersection of $W^u(S^+) \cap \Sigma$, $W^s(S^-) \cap \Sigma$

Credits: Ollè, Rodríguez, Soler, CNSNS (2017)

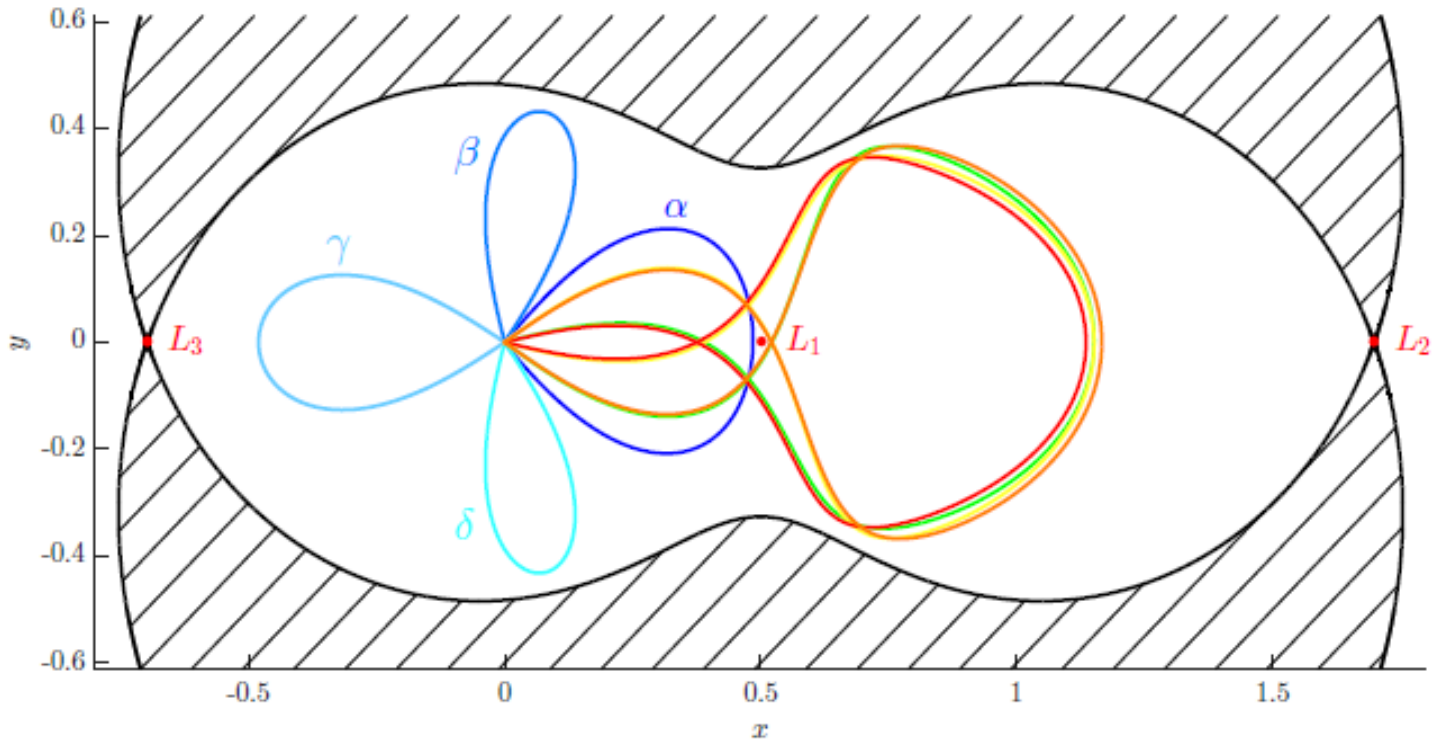
Ejection-Collision Orbit (ECO)



Second intersection of $W^u(S^+) \cap \Sigma$, $W^s(S^-) \cap \Sigma$

Credits: Ollè, Rodríguez, Soler, CNSNS (2017)

Ejection-Collision Orbit (ECO)



ECO for $\mu = 0.5$ and $h = H(L_2)$

Credits: Ollè, Rodríguez, Soler, CNSNS (2017)

Outline

Interplanetary network

Space mission design

- Patched-conic method: Keplerian approximation of a trajectory by linking different trajectories of a chain of two-body problems.
  The Sun – SC problem is followed until the SC reaches the sphere of influence of a planet.

  $$\text{Planet} - \text{Spacecraft} \longrightarrow \text{Sun} - \text{SC} \longrightarrow \text{Planet - SC} \longrightarrow \dots$$

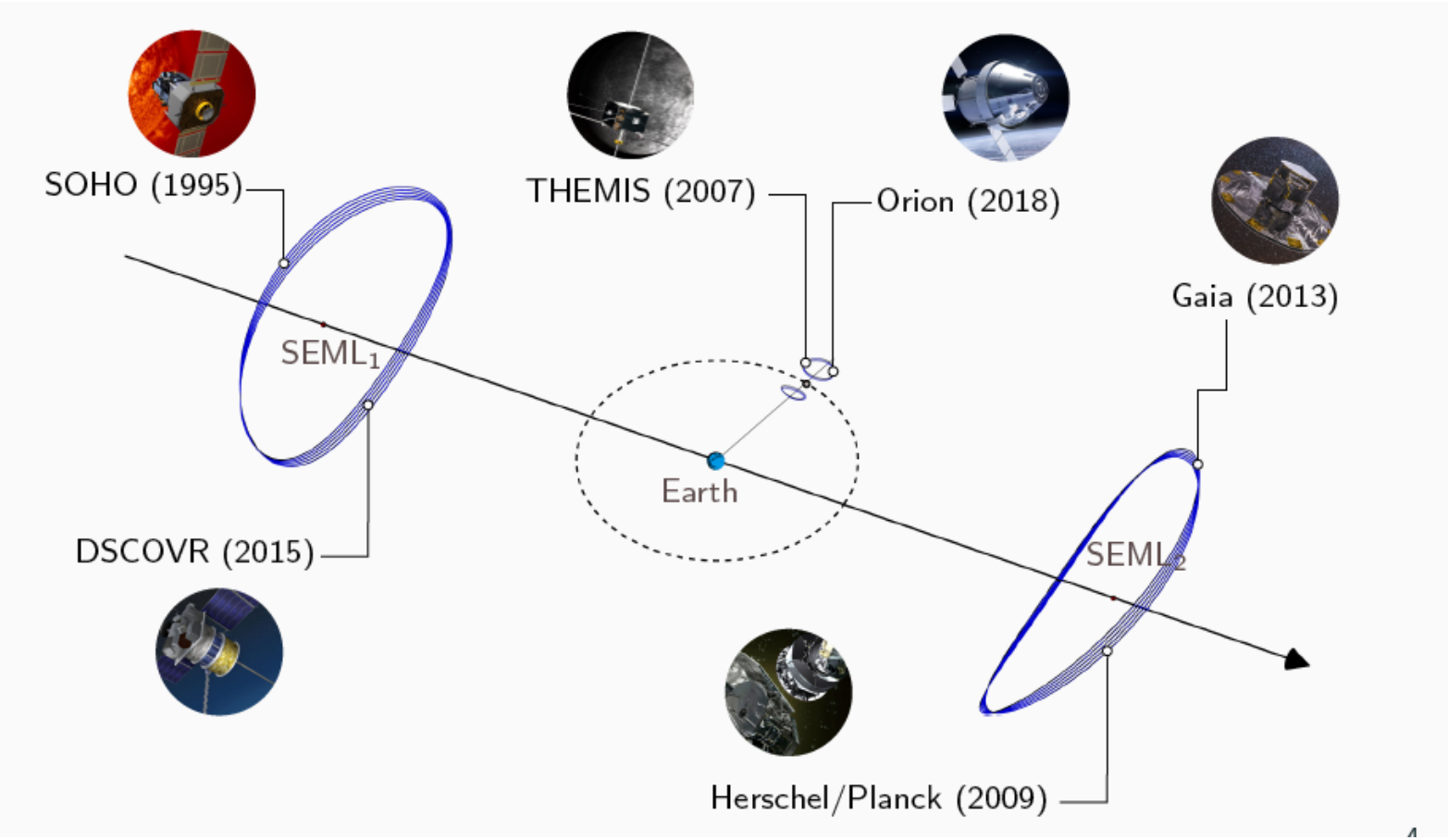
  Pioneer (1972), Voyager (1977), Galileo (1989), Mars Observer (1992)

- More demanding missions need better approximations. The Circular RTBP can be used as a (very good) first model to obtain initial seeds that can be refined to obtain solutions in a more complex models.

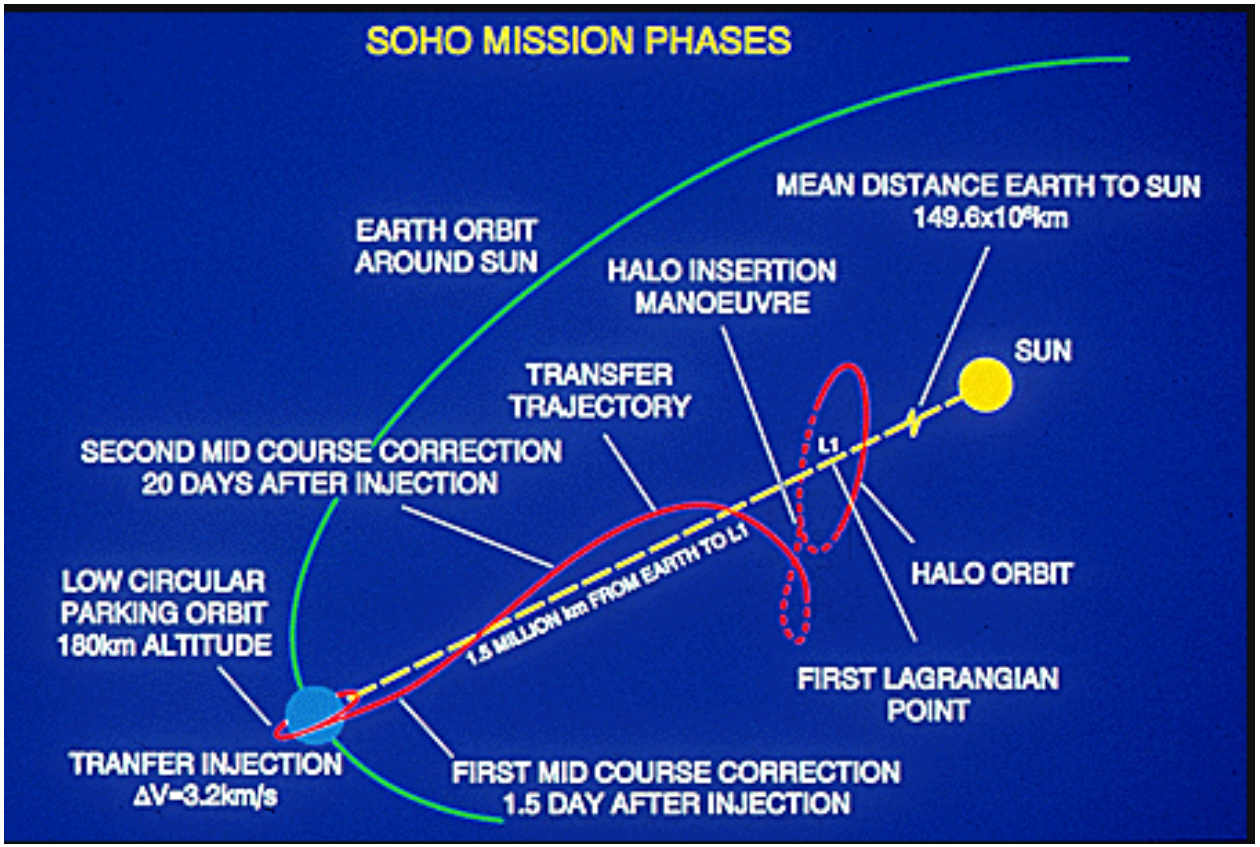
  $$\text{“Low energy transfers ”} + \Delta v$$

  Navigation using invariant manifolds and chains of homo/heteroclinic connections.

Libration point orbits missions



SOHO 1995 – ...

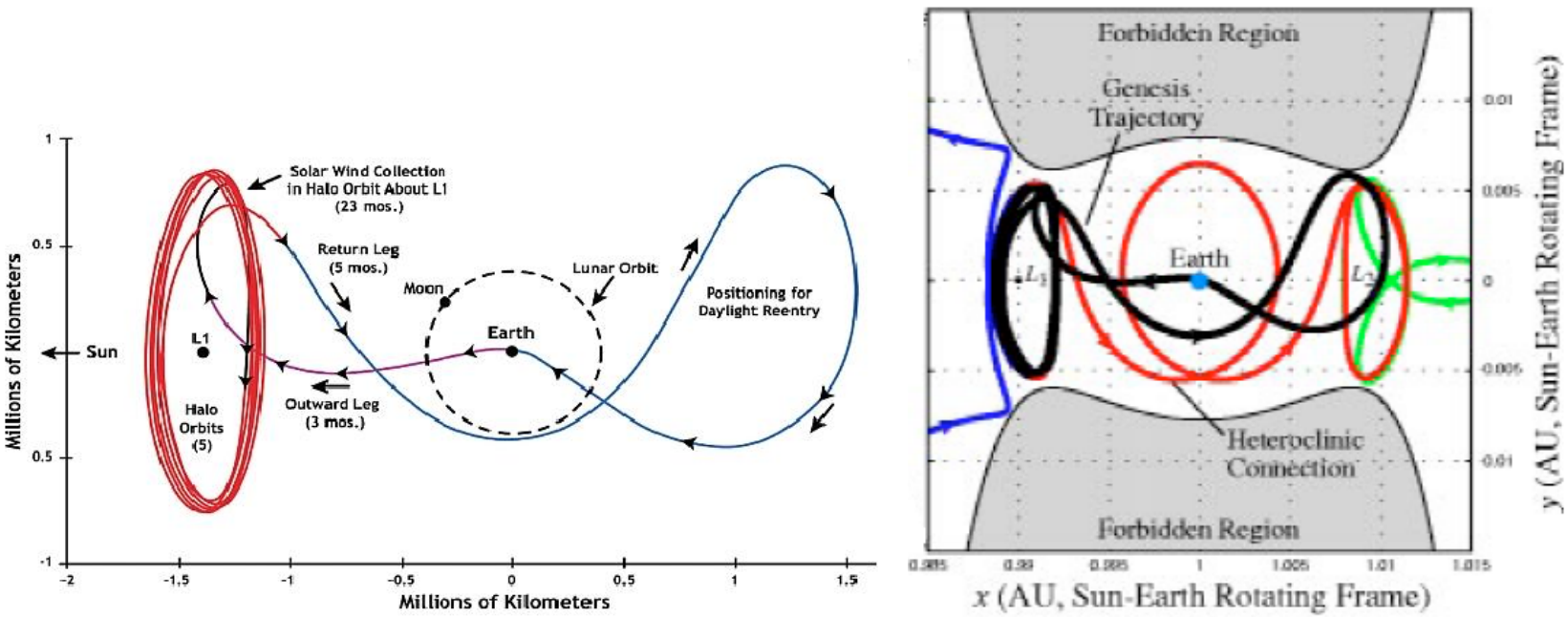


Schematic view of the SOHO transfer to a halo orbit around Sun-Earths L_1 point.

Credit: NASA

Genesis 2001-2004

Genesis is considered the first mission designed using modern Dynamical Systems theory.



First approximation chain starting at parking orbit around the Earth:

$$W^s(\Gamma_1) \to \Gamma_1 \to W^u(\Gamma_1) \cap W^s(\Gamma_2) \to W^u(\Gamma_2) \cap E$$

credits: NASA and Koon et al. Chaos (2000)

Artemis 2010

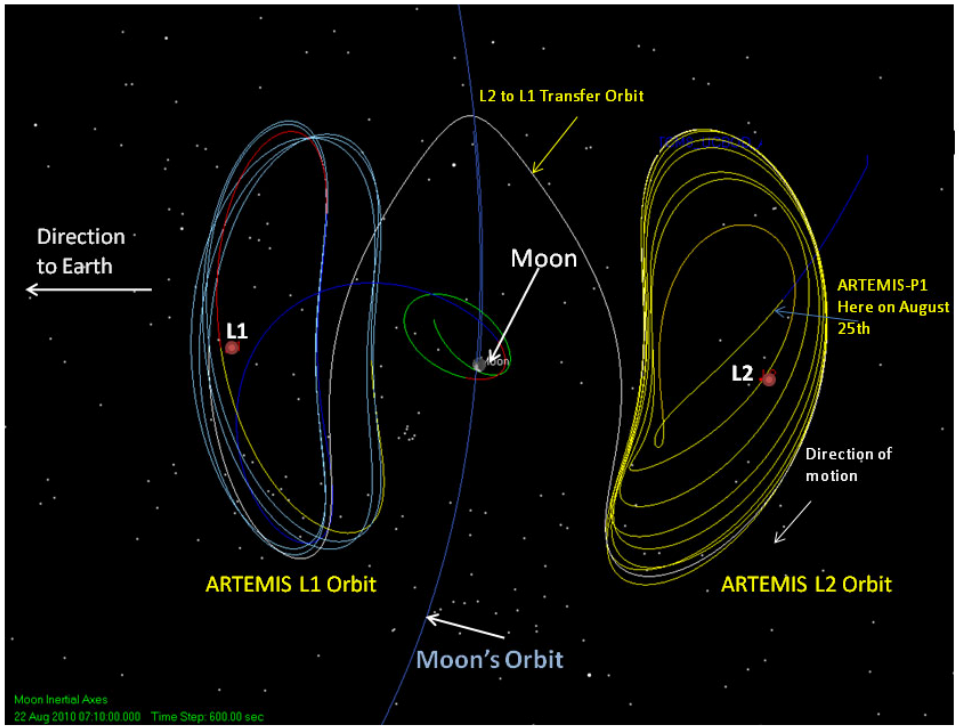
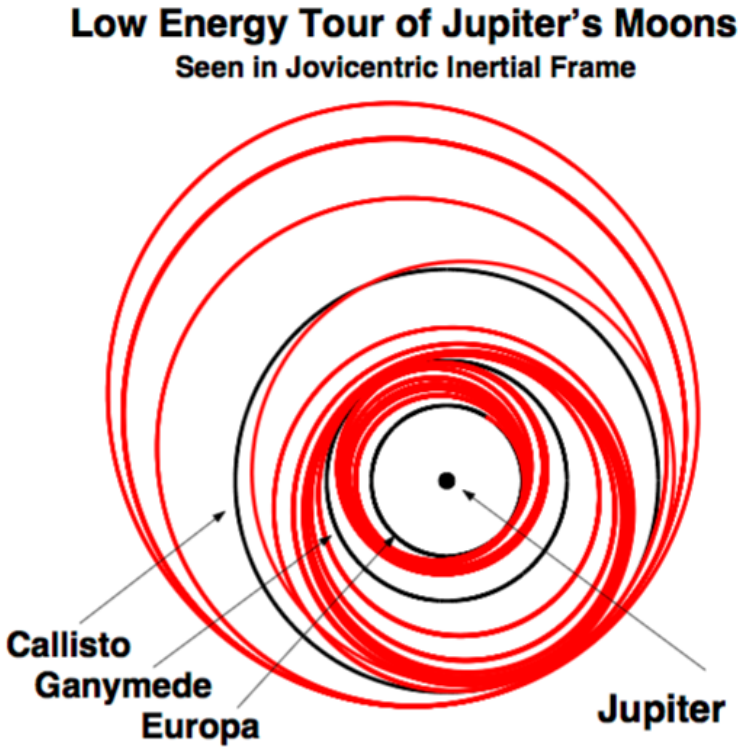


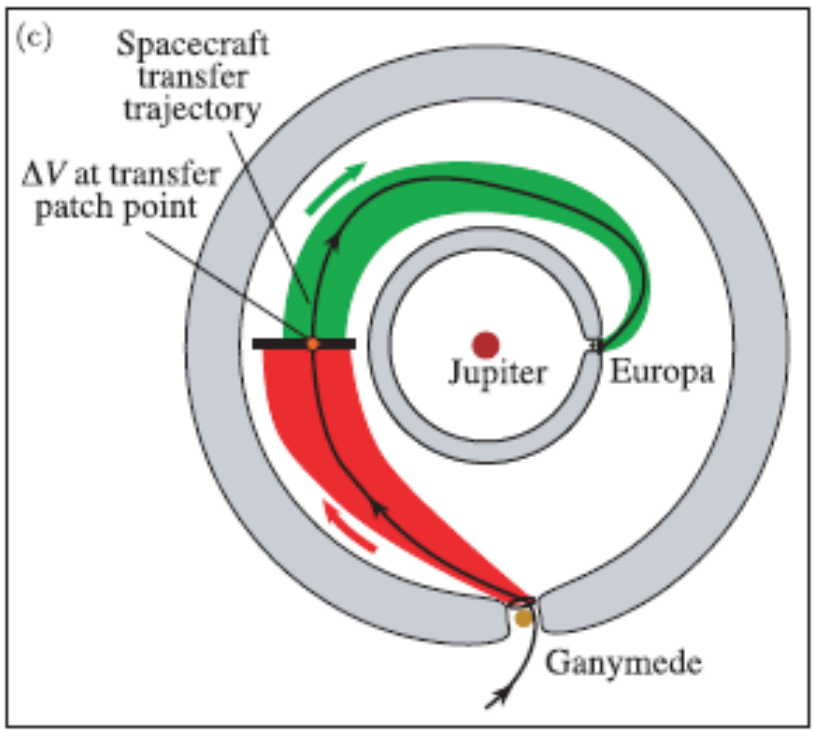
Illustration of ARTEMIS-P1 Librations Orbits Credit: NASA GSFC

Mission Themis was launched to Earth-Moon L_2 point in 2007. In 2010 two satellites were reconverted to Artemis mission: one around L_2 while the other was sent to L_1 through a heteroclinic connection.

Credit: NASA

Low energy tour on Jupiter's Moons



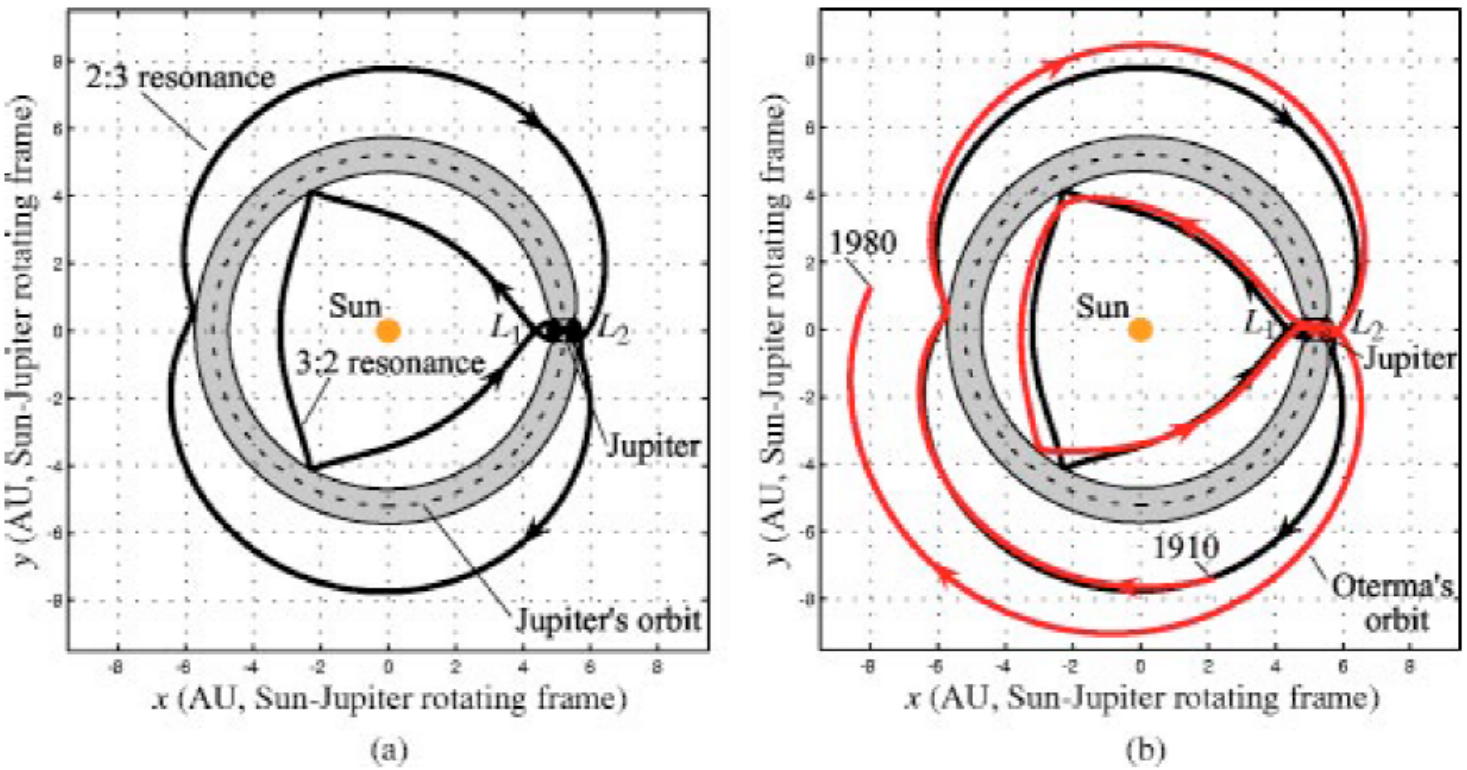


Chain of diferent CRTPB:
Transfer through $L_{1,2}$ in Jupiter-Ganymede's system – "heteroclinic connection" of i.m. of different systems – transfer through $L_{1,2}$ in Jupiter-Europas's system

Credits: Gomez et al, Nonlinearity (2004)

Oterma

Oterma follows a trajectory outside Jupiter's orbit close to 2:3 resonance, has a passage through Jupiter's region, and then follows a trajectory inside Jupiter's orbit close to 3:2 resonance



Homoclinic to L_1 + homoclinic to L_2

Credits: Koon et al, Chaos (2000)

Resonances

Resonances are defined in terms of two body dynamics:

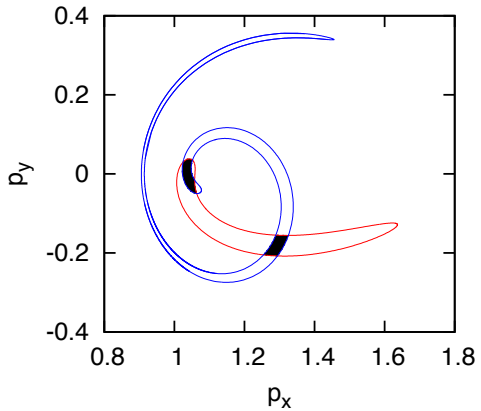
- An elliptic (keplerian) orbit is $p:q$ resonant with Jupiter, if it performs p revolutions around the Sun while Jupiter performs q revolutions.
- The mean motion equals $a^{-3/2} = p/q$, being a the semimajor axis that can be calculated as

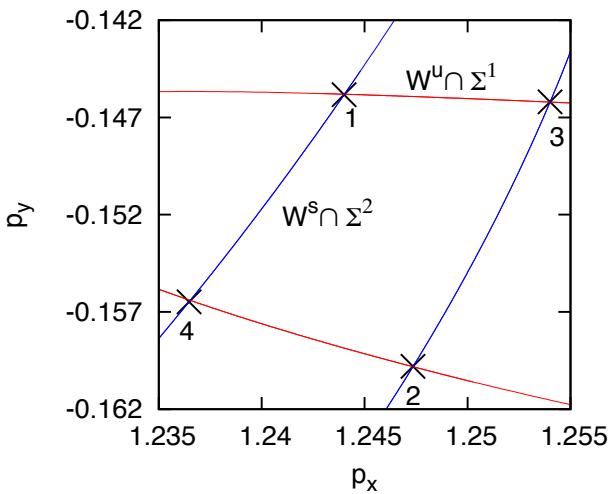
$$a^{-1} = \frac{2}{r} - v^2$$

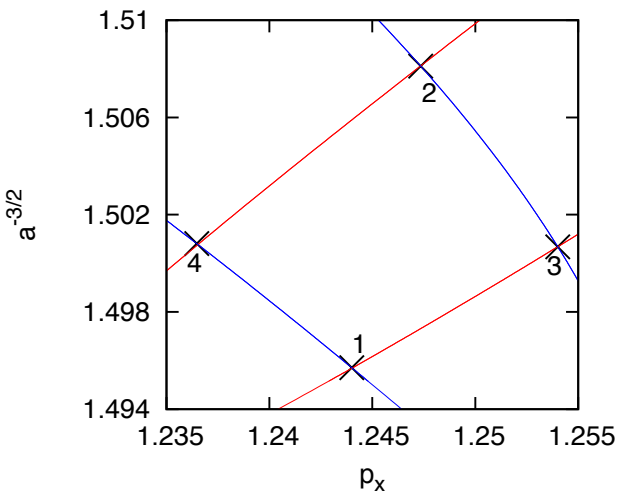
- For trajectories of the CRTBP that behave essentially as a two-body solution, a will be approximately constant. We can compute its approximate value for the the homoclinics that provide the dynamical chains

Families of homoclinic connections: resonances

Homoclinic connections to OL_1 in the Sun-Jupiter CRTBP: $W^s_+(OL_1) \cap W^u_+(OL_1)$



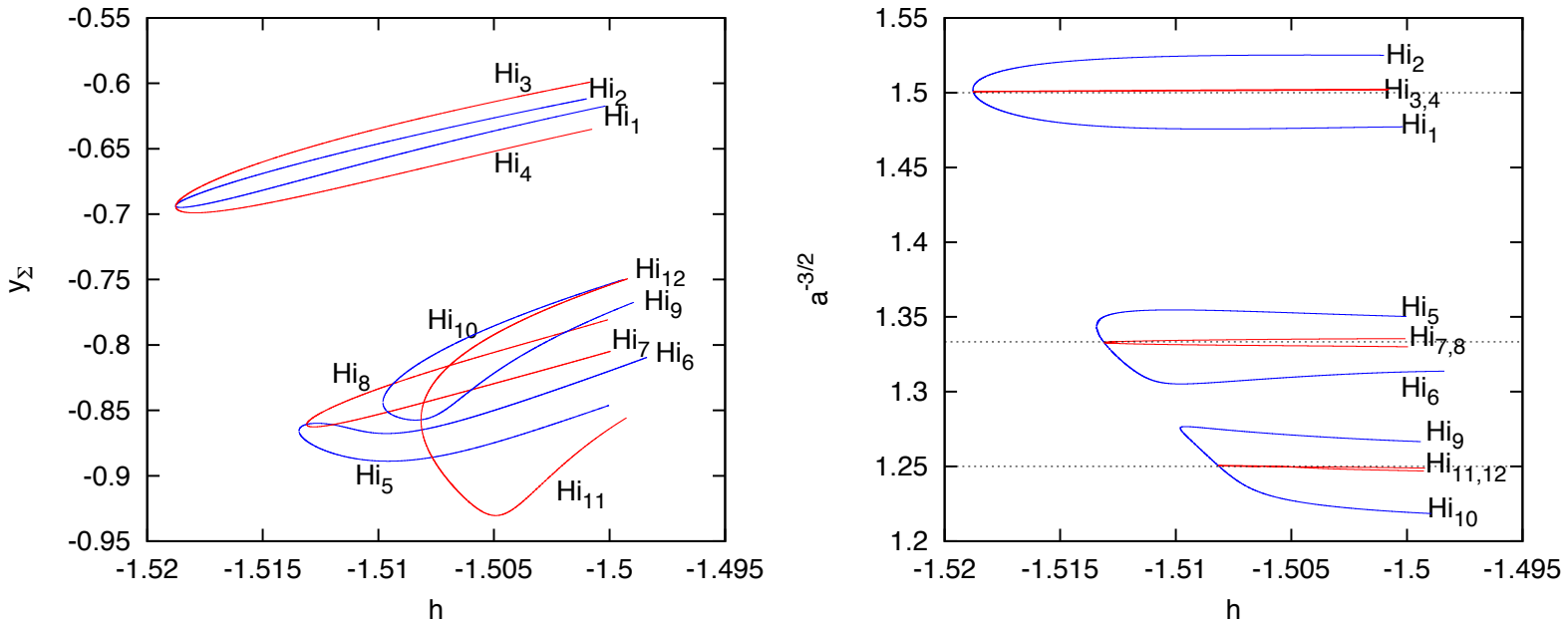




Credits: B, Mondelo, Ollè, Nonlinearity (2013)

Families of homoclinic connections: resonances

Families of inner homoclinic connections

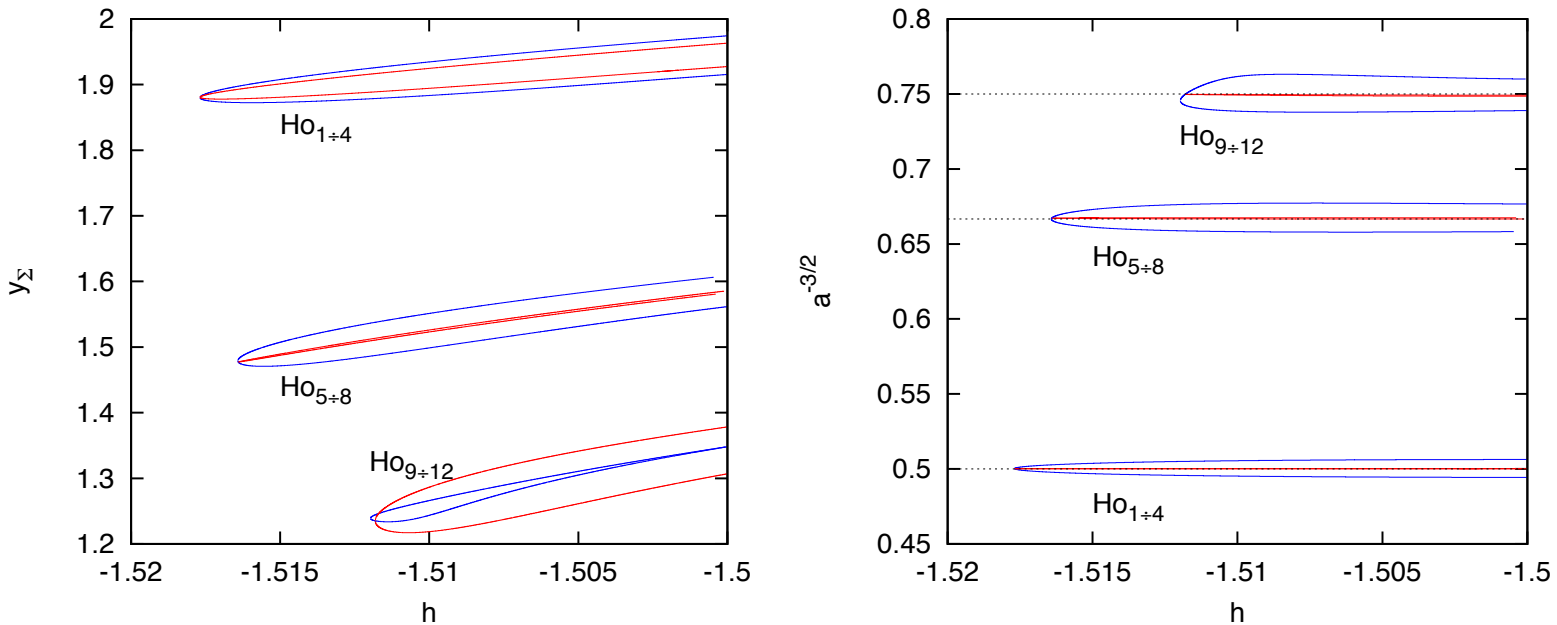


Resonances at 3:2, 4:3, 5:4

Credits: B, Mondelo, Ollè, Nonlinearity (2013)

Families of homoclinic connections: resonances

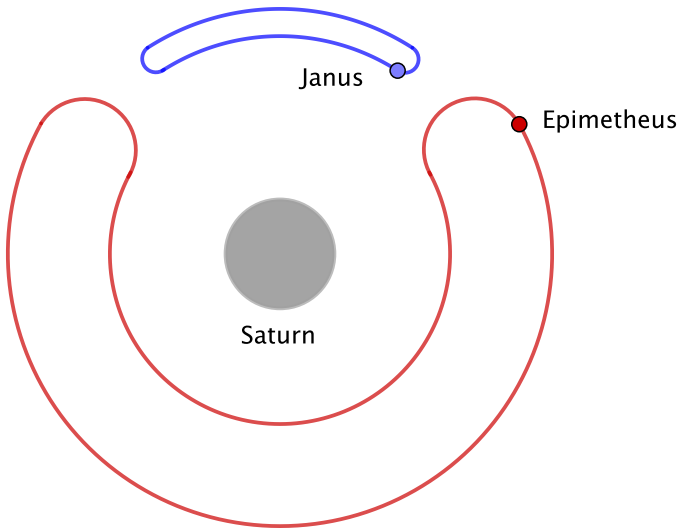
Families of outer homoclinic connections



Resonances at 3:4, 2:3, 1:2

Credits: B, Mondelo, Ollè, Nonlinearity (2013)

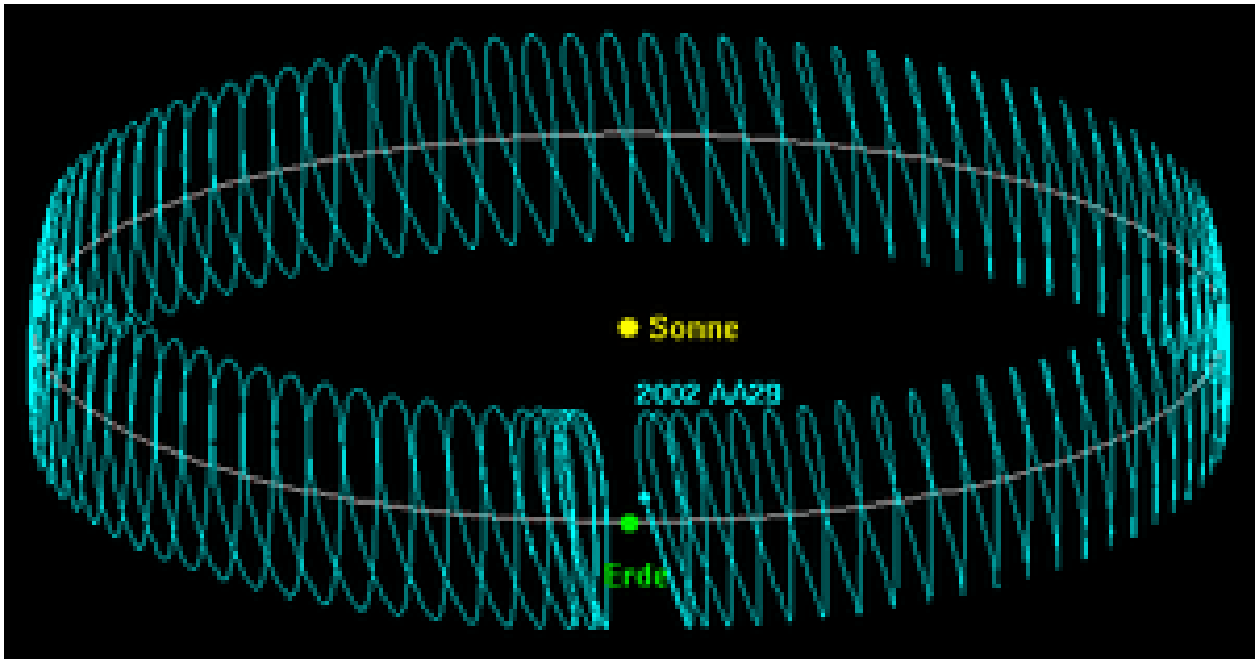
Co-orbital motions



Schematic motion of Janus and Epimetheus around Saturn. Each satellite is shown in a rotating frame.

Credits: NASA-JPL

Co-orbital motions

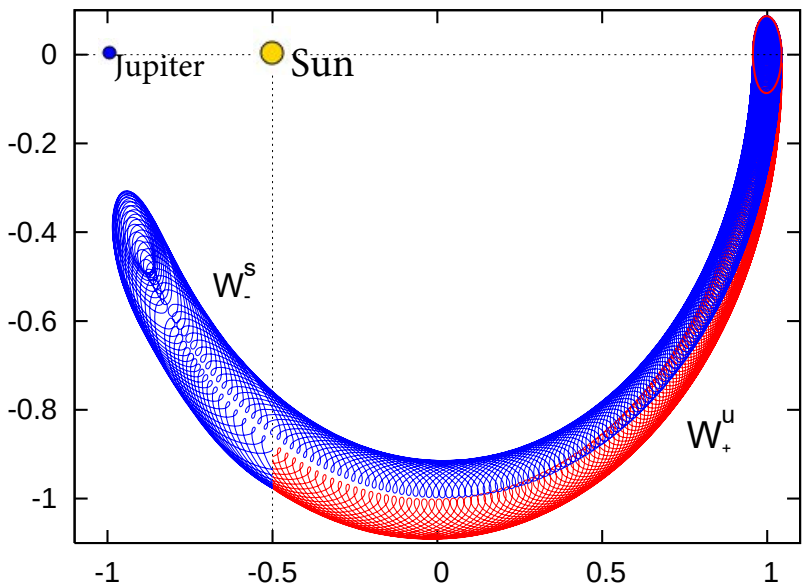


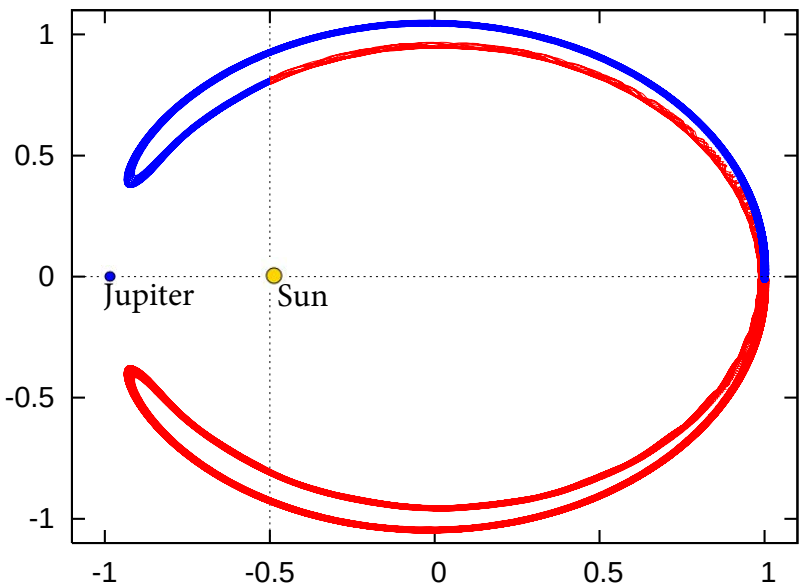
Schematic motion of Earth quasi-satellite 2002 AA29. Co-orbital (horseshoe) motion in Sun-Earth system.

Credits: JPL

Homoclinics to Lyapunov orbits around L_3

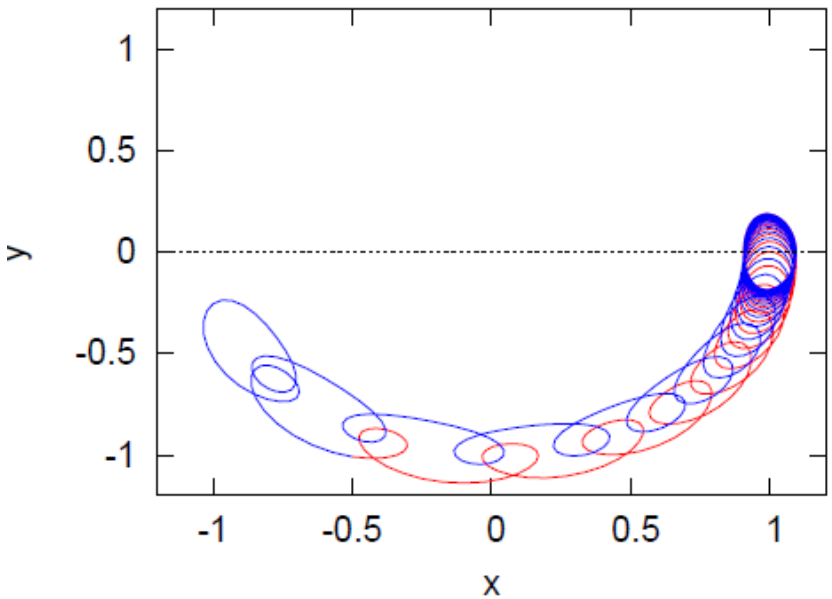
Sun - Jupiter CRTBP - (half) horseshoe-shape homoclinics



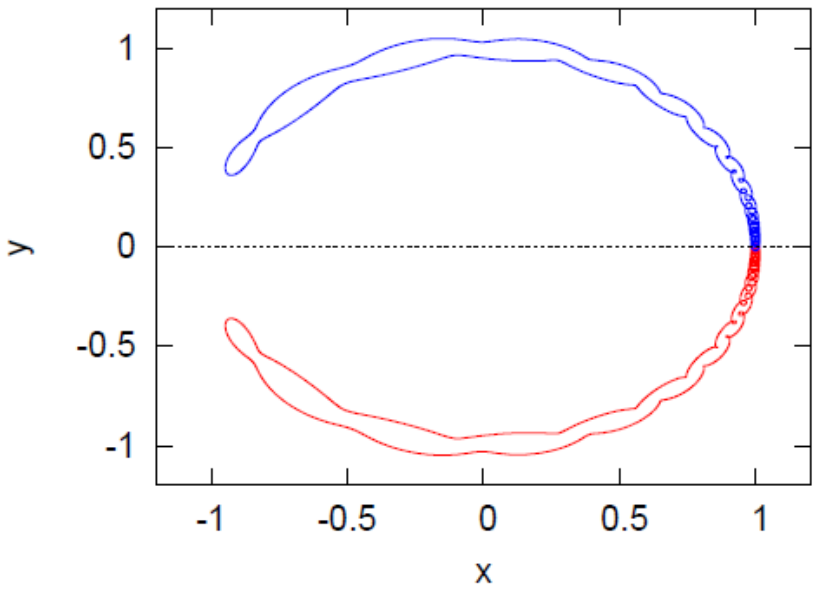


Homoclinics to Lyapunov orbits around L_3

Sun - Jupiter CRTBP - (half) horseshoe-shape homoclinics



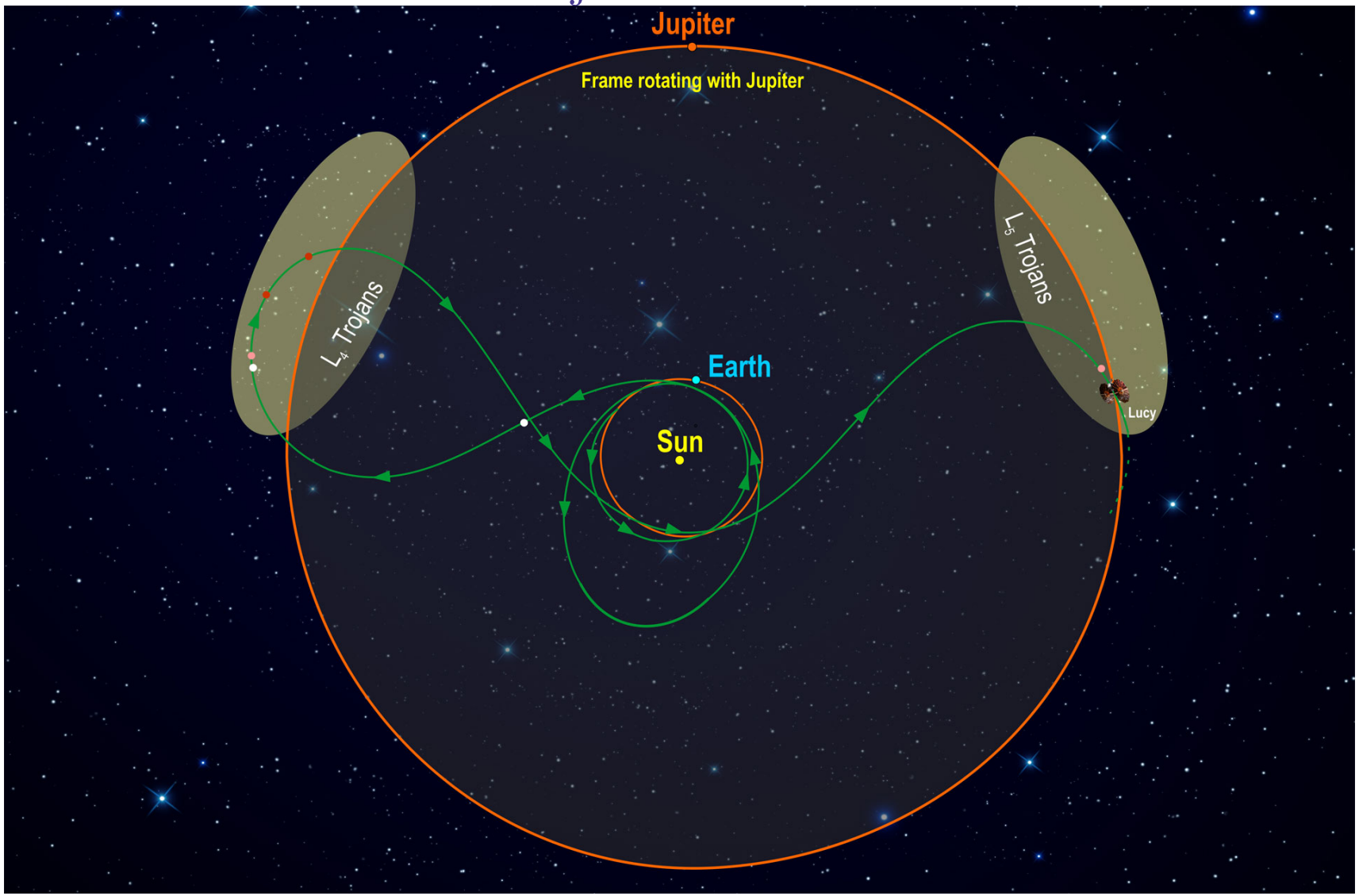
$h = -1.49602677$



$h = -1.50035535$

Horseshoe orbits are organized in families from which vertical (3d) horseshoe orbits bifurcate

Trojan Asteroids



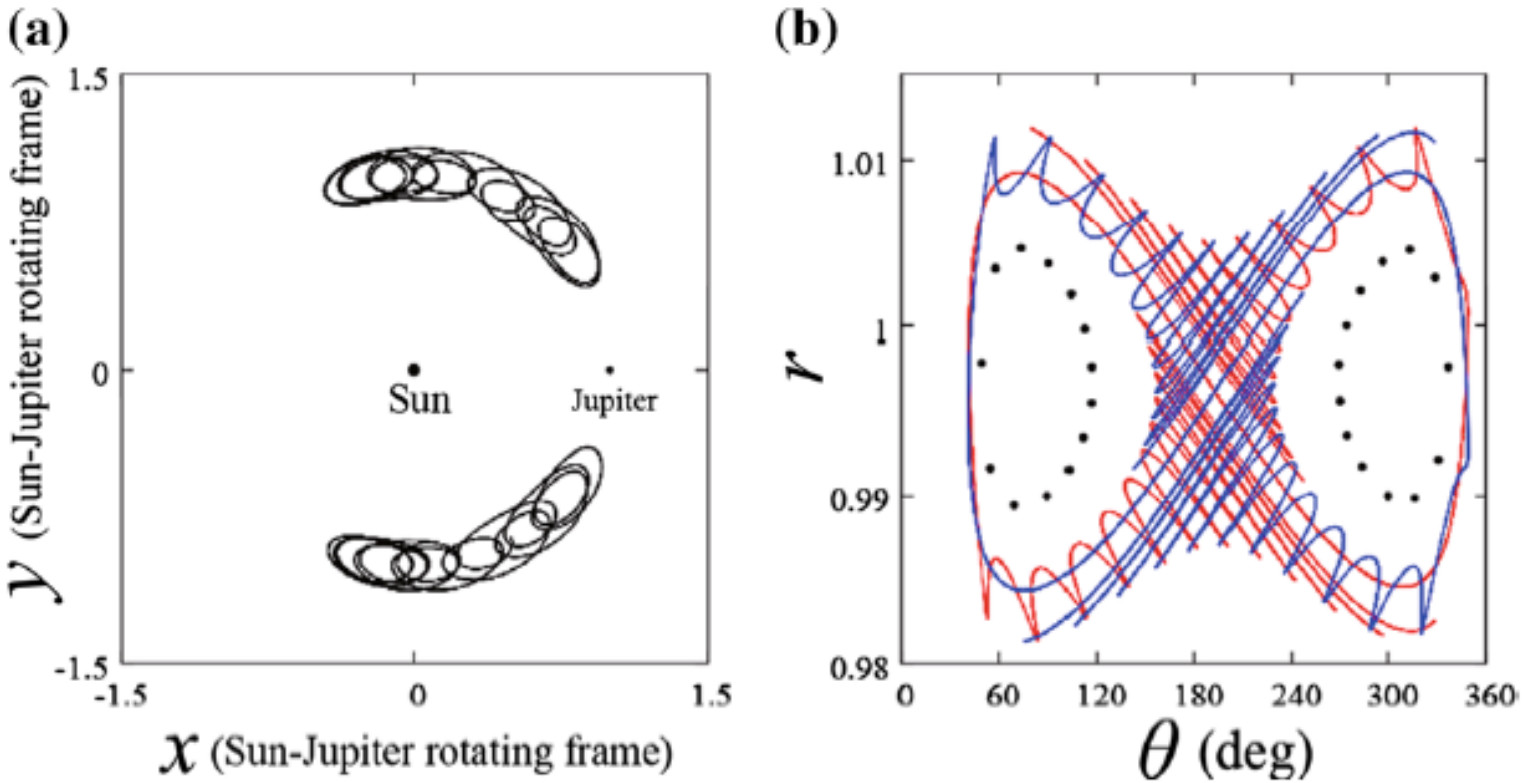
Credits: NASA (2018)

Regions of effective stability around triangular equilibrium points

- The triangular equilibrium points $L_{4,5}$ are linearly stable for $\mu < \mu_R \simeq 0.0381....$ Using KAM theory, $\exists$ a dense set of invariant tori close enough to the equilibria.
- In the planar case, for each fixed energy level, the KAM tori of dimension 2 act as barriers for the dynamics. This is not true for the spatial case.
- Numerical simulations provide evidences of large regions of practical stability near $L_{4,5}$ in which the motion is non-stable but initial conditions take a long time to escape
- There is numerical evidence on the role of the centre-unstable and centre-stable manifolds of the collinear point L_3

Jumping Jupiter's Trojans

Sun–Jupiter RTBP. Poincaré section $\dot{r} = 0$ (Sun centered polar coord. (r, θ))

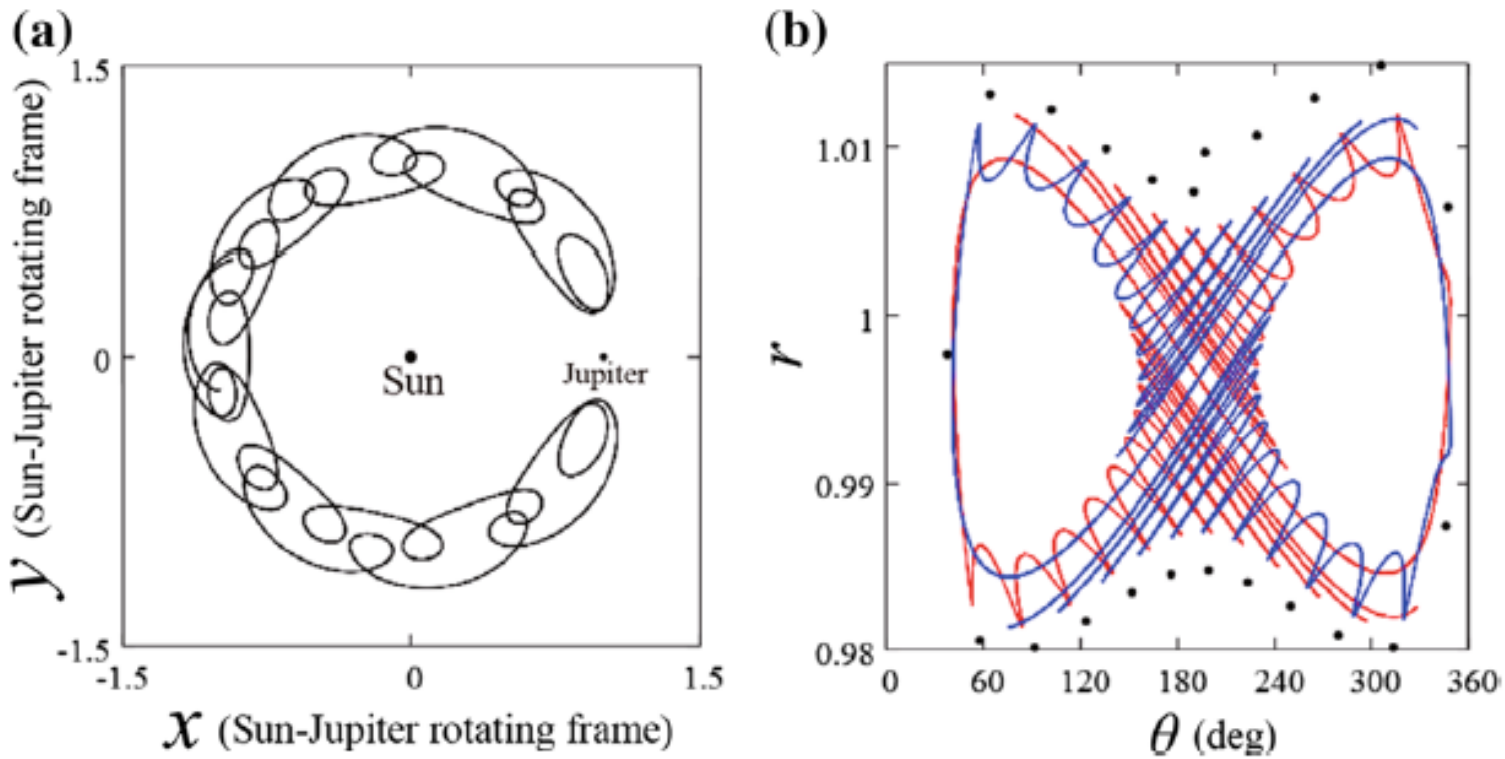


Tadpole orbits and $W^{u/s}(OL_3)$ (Lyapunov planar orbit around L_3)

Credits: Oshima, Yanao, Cel.Mech.Dyn.Astr. (2015)

Jumping Jupiter's Trojans

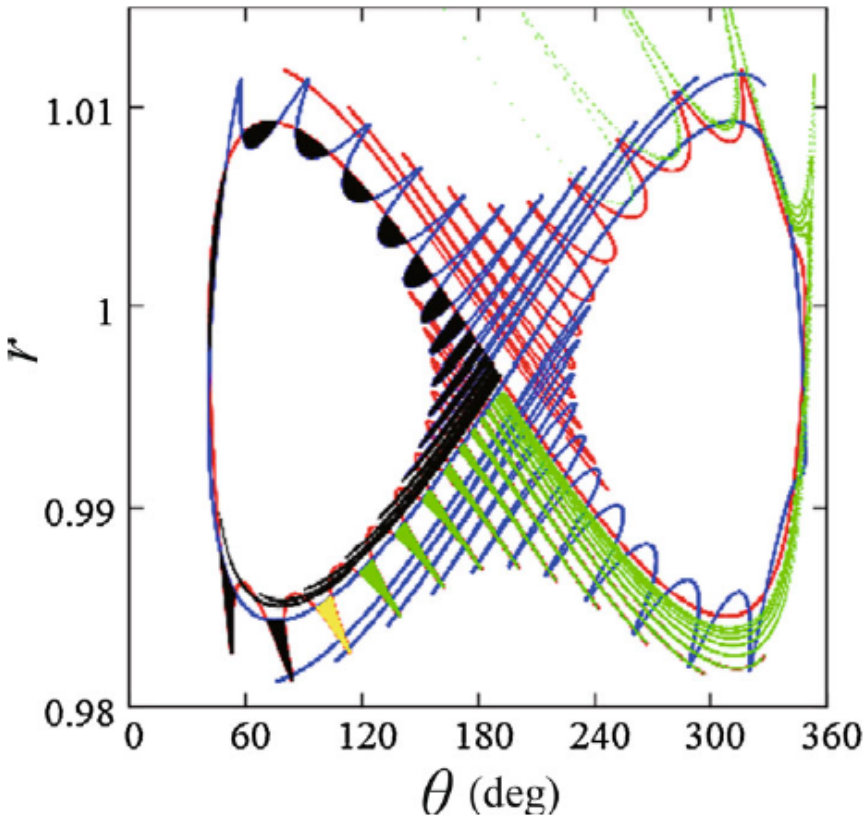
Sun–Jupiter RTBP. Poincaré section $\dot{r} = 0$ (Sun centered polar coord. (r, θ))



Horseshoe orbit and $W^{u/s}(OL_3)$ (Lyapunov planar orbit around L_3)

Credits: Oshima, Yanao, Cel.Mech.Dyn.Astr. (2015)

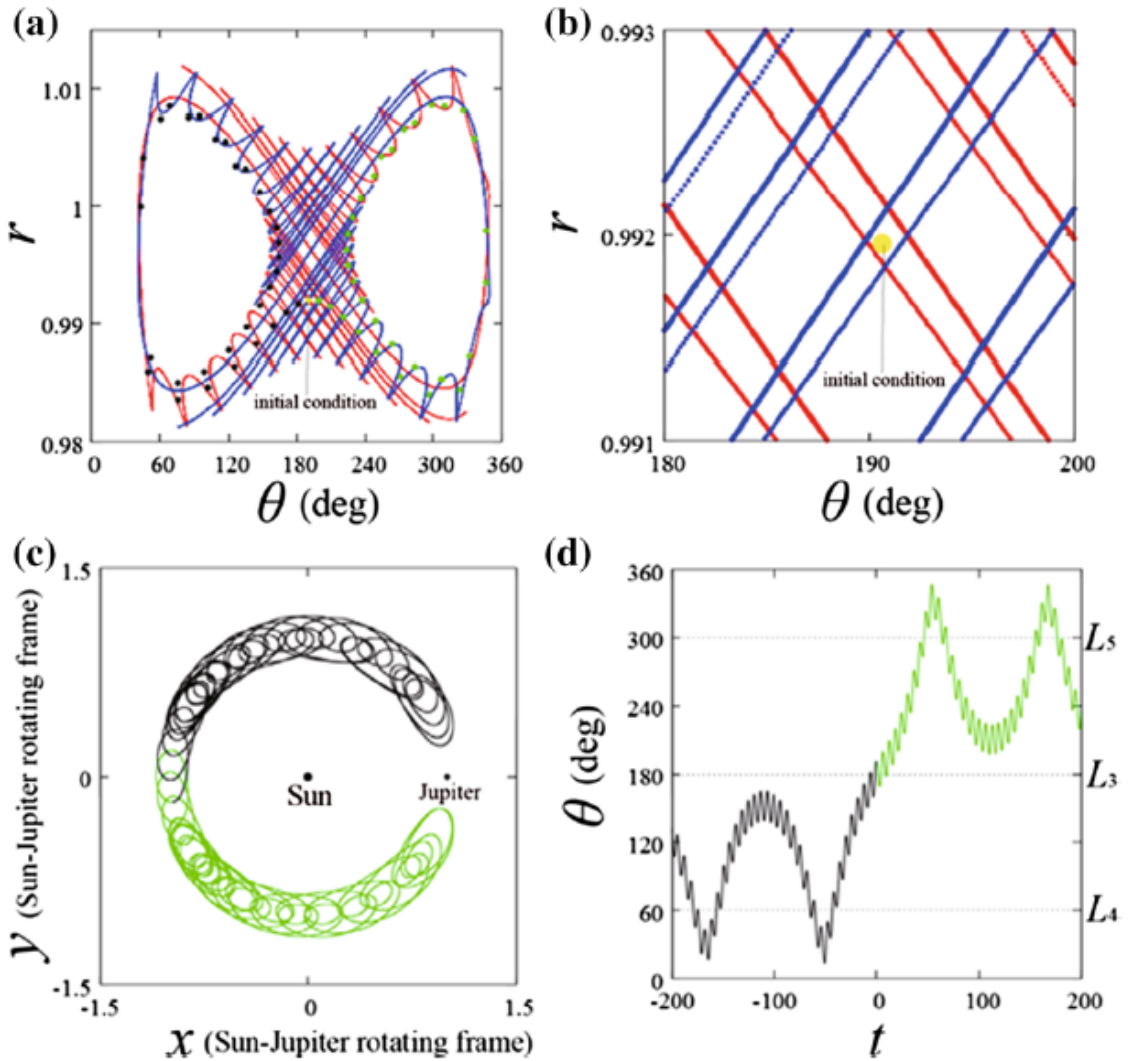
Jumping Jupiter's Trojans



Regions of escape from and capture into (forwards or backwards) the vicinity of L_4 through a neighborhood of L_3

Credits: Oshima, Yanao, Cel.Mech.Dyn.Astr. (2015)

Jumping Jupiter's Trojans



Credits: Oshima, Yanao, Cel.Mech.Dyn.Astr. (2015)

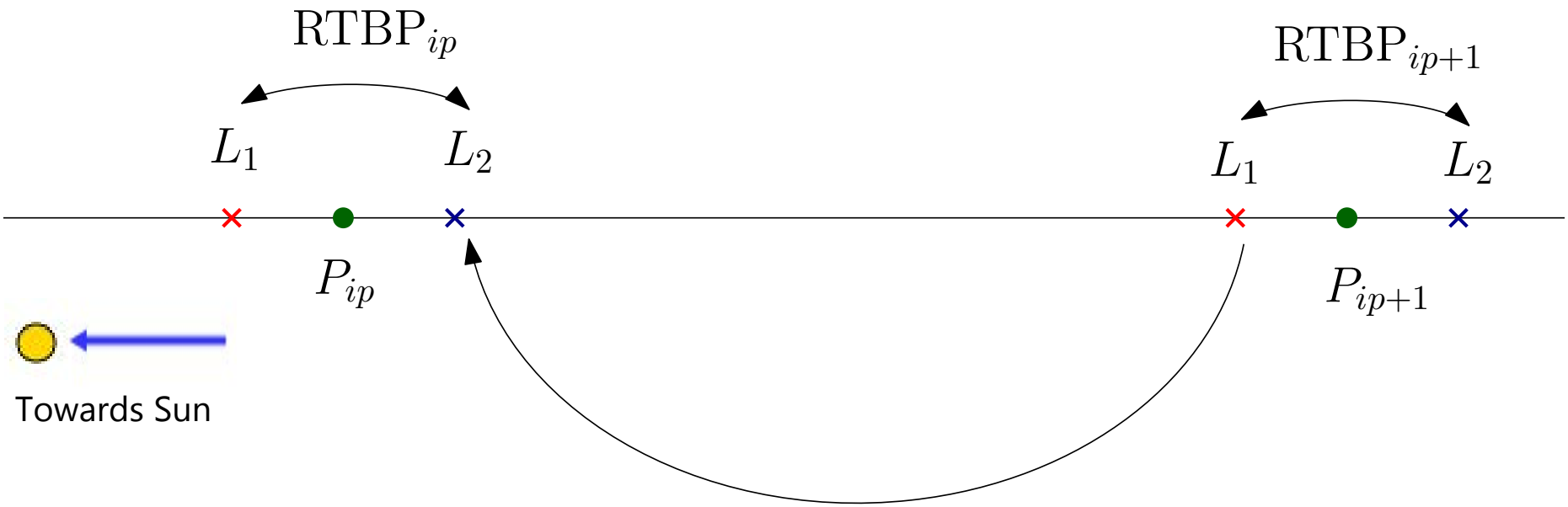
Outline

Remember that all models are wrong; the practical question is how wrong do they have to be to not be useful.

George E. P. Box (1918–2013)

Short-time mass transport

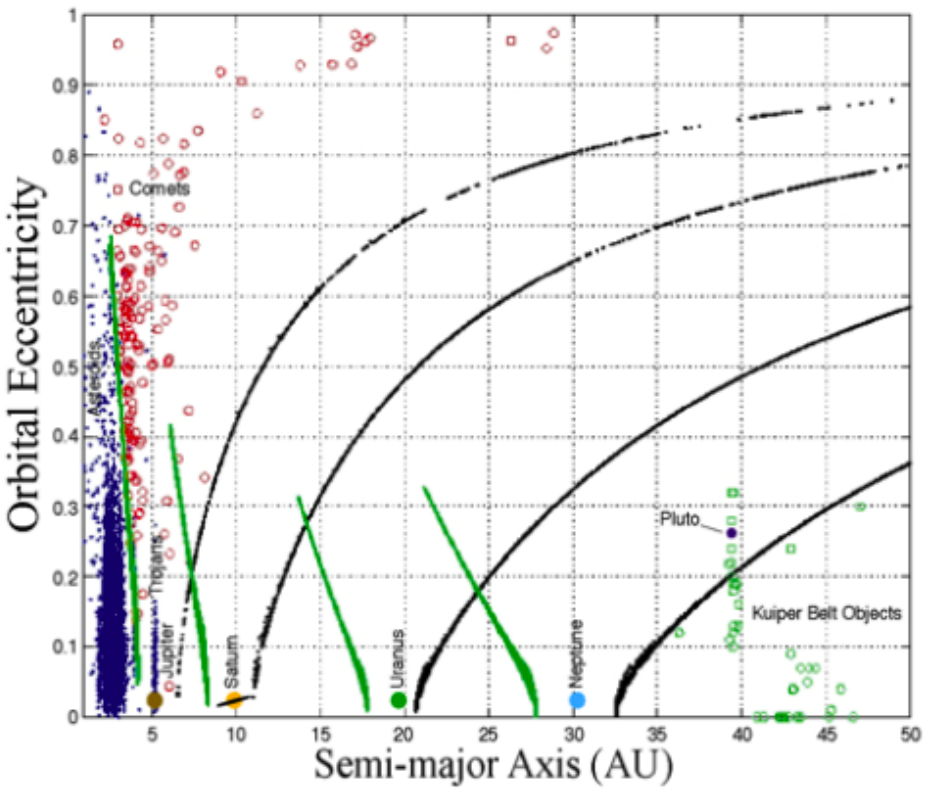
Short-time transport is based on the ∃ of pseudo-heteroclinic connections between libration point orbits of uncoupled pairs of consecutive CRTBP



Scheme of the chain of connections between two consecutive Sun–Planet CRTBP

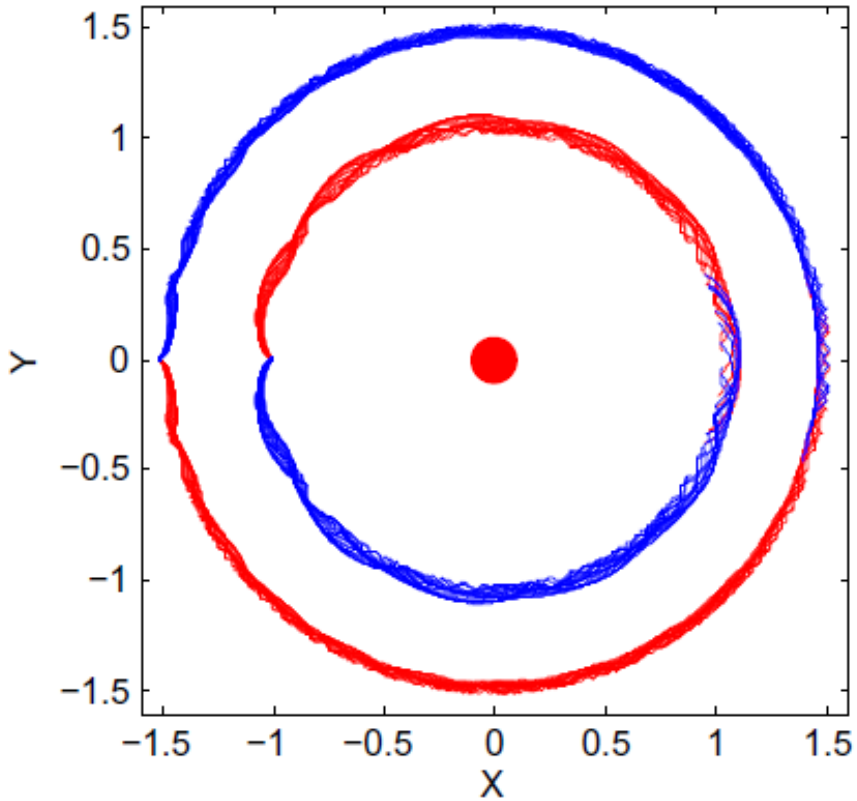
Short-time mass transport

Osculating orbital elements of the 1-dimensional invariant manifolds $W^u(L_2)$ and $W^s(L_1)$ of the collinear libration points L_1 and L_2 in a chain of CRTBP



Credits: Lo, Ross, Tech. Report. (1999)

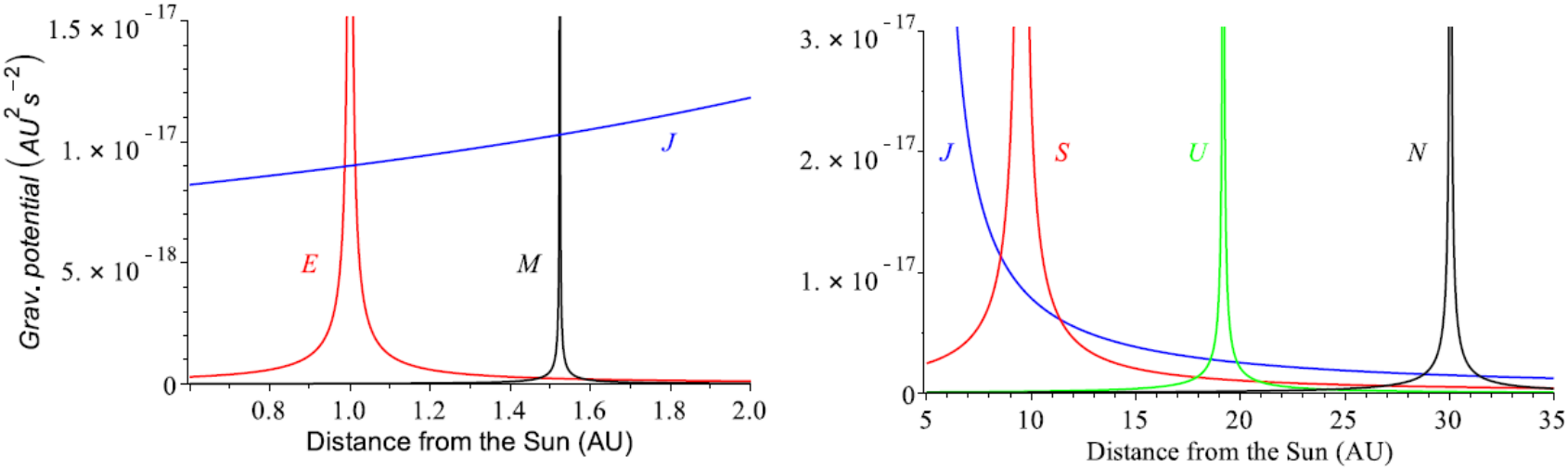
Short-time mass transport



Branches of stable (blue) and unstable (red) invariant manifolds of p.o. around L_1 in the Sun–Mars and L_2 in the Sun–Earth CRTBP. In "short-time", they do not intersect.

Credits: Y.Ren, M.Masdemont, G. Gómez, E.Fantino, Commun. Nonlinear Sci Numer. Simulat. (2012)

Gravity potential in the Solar System

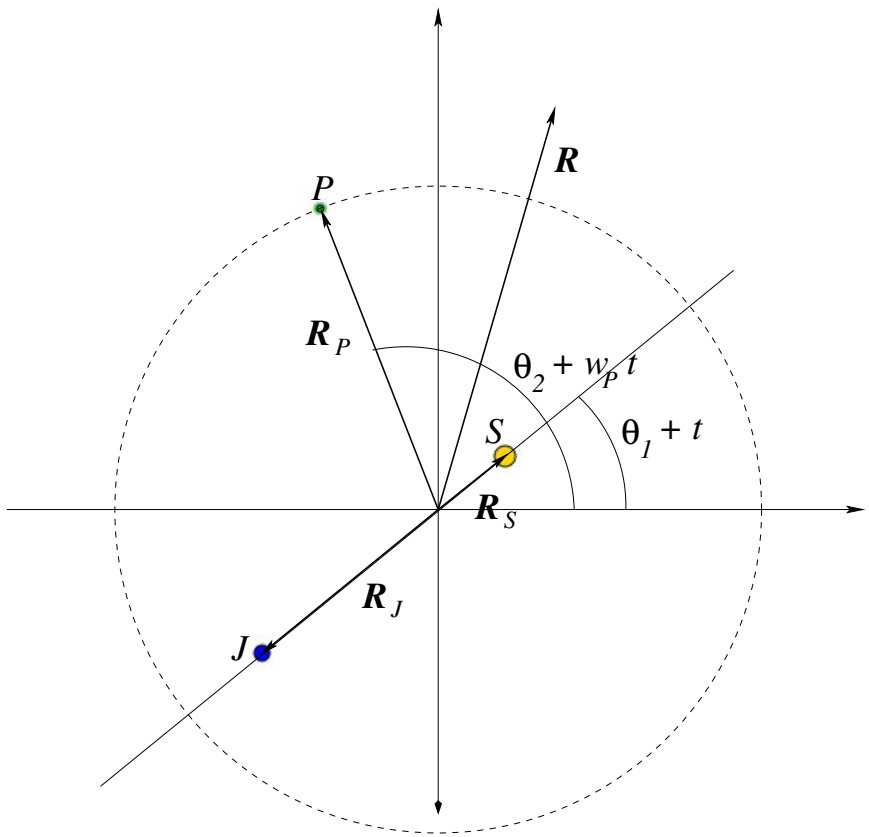


Values of the gravity potential (in $AU^2 s^{-2}$) of the: (LEFT) Earth, **Mars** and Jupiter up to 2 AU and (RIGHT) Jupiter, Saturn, Uranus and **Neptune** in the outer Solar System

Bicircular Restricted Four Body Problem (BR4BP)

Is a periodic perturbation of the CRTBP that includes the gravitational effect of a third primary on the particle.

Sun – Jupiter – Planet – particle



Bicircular Restricted Four Body Problem (BR4BP)

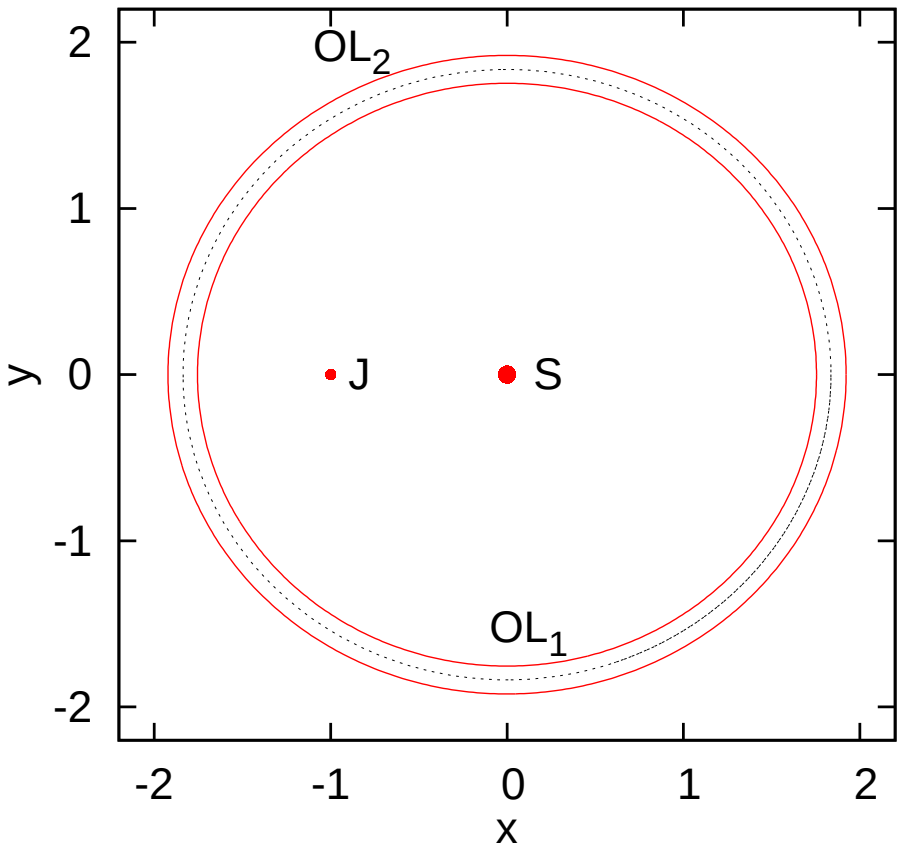
Is a periodic perturbation of the CRTBP that includes the gravitational effect of a third primary on the particle.

$$H = \frac{1}{2}(p_x^2 + p_y^2 + p_z^2) - xp_y + yp_x - \frac{1-\mu}{r_1} - \frac{\mu}{r_2} - \frac{\mu_P}{r_P} + \frac{\mu_P}{a_P^2}(y\sin\theta + x\cos\theta)$$

$$\begin{aligned} r_1 &= ((x-\mu)^2 + y^2)^{1/2}, \\ r_2 &= ((x-\mu+1)^2 + y^2)^{1/2}, \\ r_P &= ((x - a_P\cos\theta)^2 + (y - a_P\sin\theta)^2)^{1/2}, \\ \theta &= \theta_2 - \theta_1 + t(\omega_P - 1), \end{aligned} \qquad \begin{aligned} \mu &= \tfrac{m_J}{m_J+m_S}, \\ \mu_P &= \tfrac{m_P}{m_J+m_S} \\ \\ \omega_P^2 a_P^3 &= 1 + \mu_P \end{aligned}$$

Dynamical substitutes in the BR4BP

The equilibrium points L_i of the Sun-Planet CRTBP are replace by periodic orbits.



Projection in configuration space (rotating coordinates) of the dynamical substitutes OL_1 and OL_2 (orbit of the planet in a dotted line)

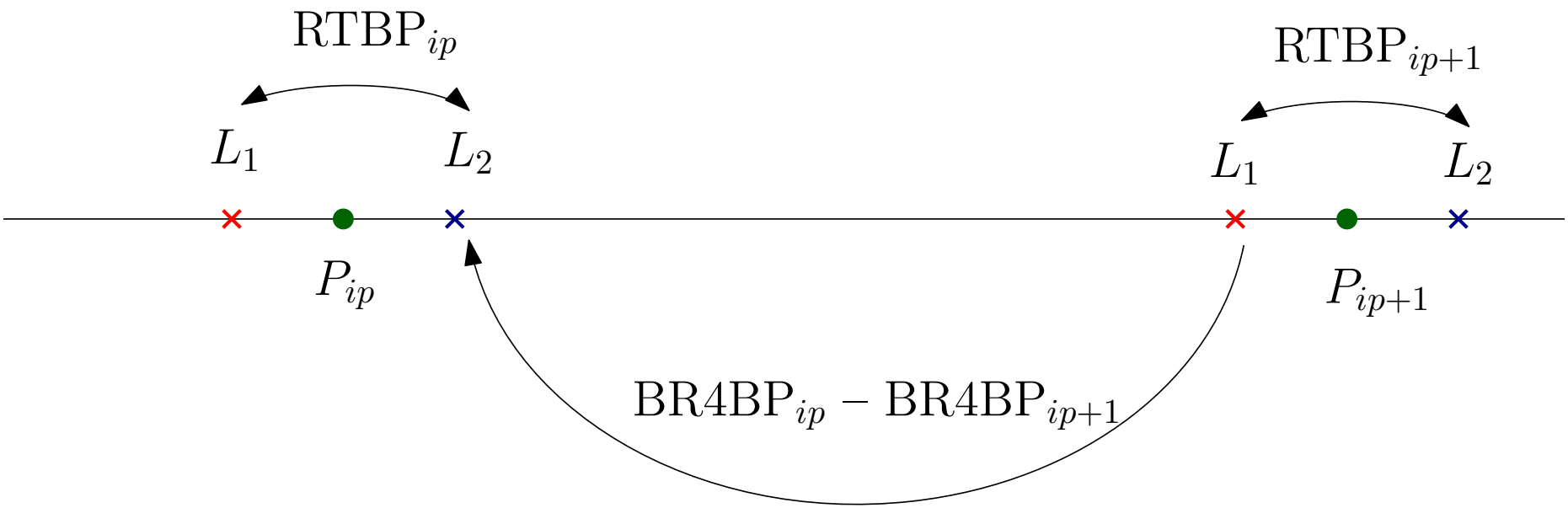
Dynamical substitutes in the BR4BP

The dynamical substitutes OL_i inherit the hyperbolicity.

Outer planets	$\Lambda(OL_i)$, $i=1,2$	Inner planets	$\Lambda(OL_i)$, $i=1,2$
Neptune	3.492, 3.286	Mars	9×10^7, 2.5×10^8
Uranus	14.105, 12.473	Earth	2.8×10^7, 3.4×10^7
Saturn	6.5×10^4, 2.5×10^4	Venus	1.5×10^7, 1×10^7

Value of the eigenvalue $\Lambda > 1$ corresponding to the dynamical substitutes OL_i^{ip}, $i=1,2$ of each bicircular problem.

Mass transport: chain of BR4BP

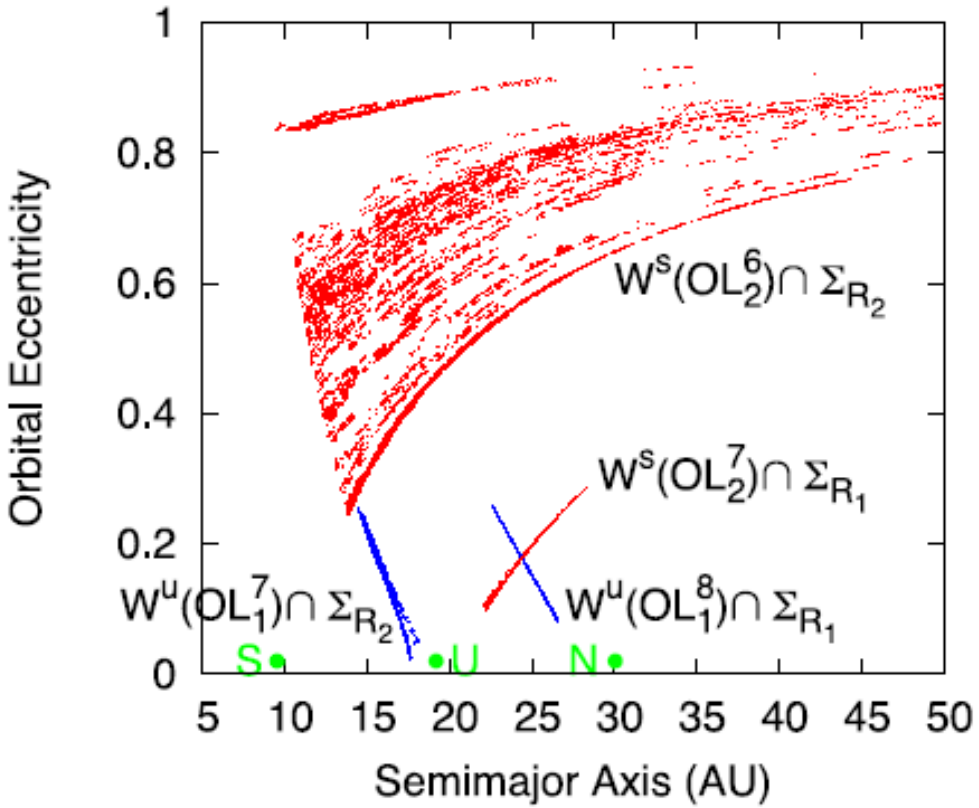


Chain of the models: Sun-Jupiter-Planet$_{ip}$ – Sun-Jupiter-Planet$_{ip+1}$

$$W^s(OL_2^{ip}) \cap W^u(OL_1^{ip+1})$$

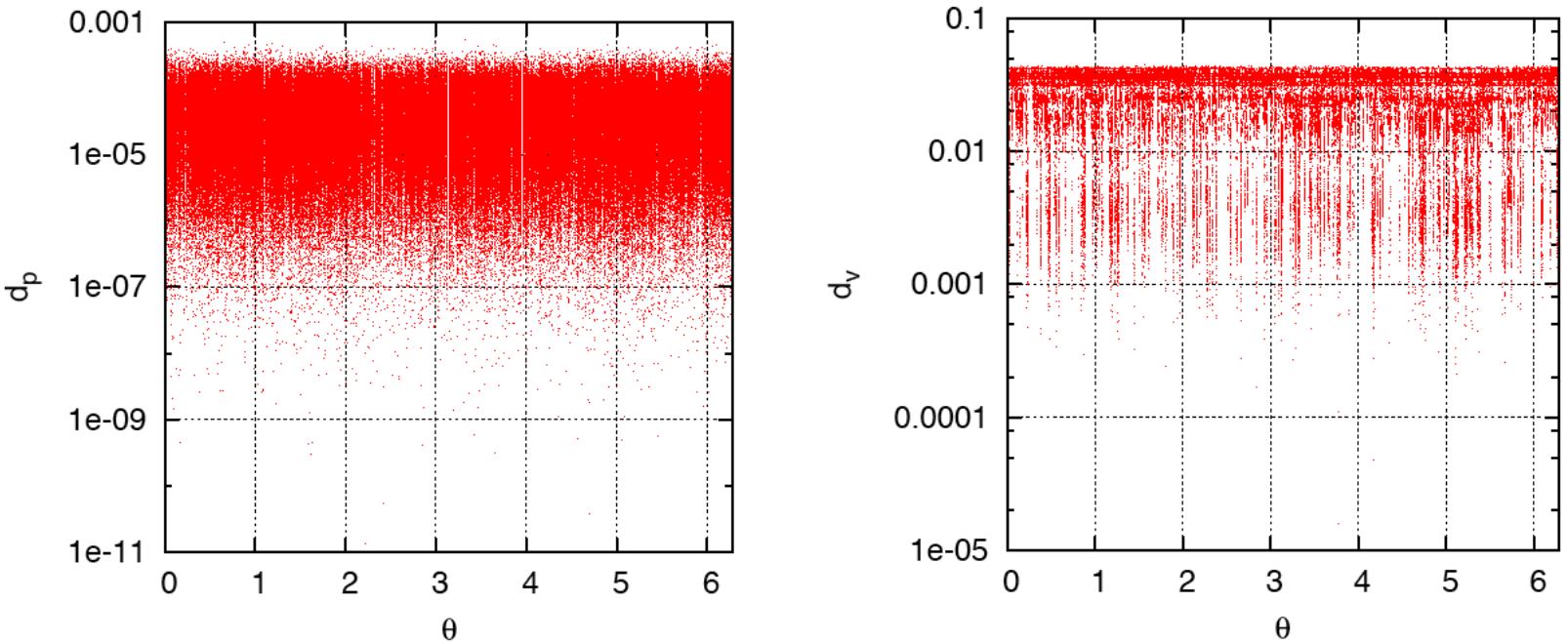
Mass transport: chain of BR4BP

Osculating semimajor axis vs eccentricity for the points (at a given section) of $W^u(OL_1^8)$ and $W^u(OL_1^7)$ (in blue), and $W^s(OL_2^7)$ and $W^s(OL_2^6)$ (red)



Mass transport: chain of BR4BP

The invariant manifolds $W^s(OL_2^{ip})$ and $W^u(OL_1^{ip+1})$ are followed up to a intermediate section R =const. to look for intersections. When there is a match in position, Δv is measured



Minimum distances d_p (positions, left) and d_v (velocities, right) between points of the invariant manifolds at the section Σ_{R_1} of the Uranus and Neptune bicircular problems

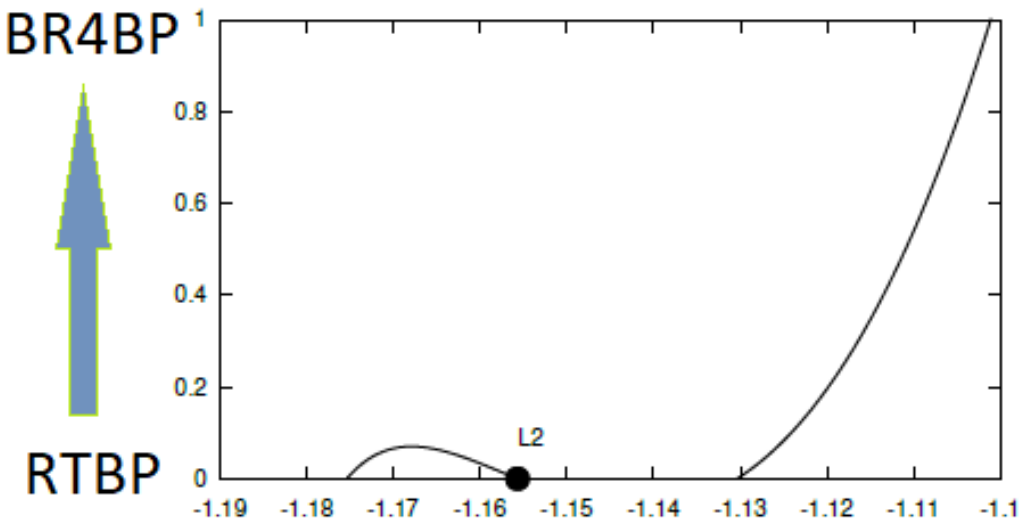
The BR4BP: yes and no

- The BR4BP can be a good model
  - to explore the intersection of the $W^{u/s}(OL_{1,2})$ in a chain of bicircular models for the outer Solar System to explain mass transport.
    Only linear approximation of $W^{u/s}$ and short time (10 000 $\times$ period Jupiter) has been taken into account. Better approximations and longer times could improve the results and give intersections, even for the inner planets.

  - for the study of the vicinity of the triangular points $L_{4,5}$ of the main two bodies (two regions of effective stability).

The BR4BP: yes and no

- The BR4BP is not useful for the study of the the translunar point in the Earth-Moon-Sun-particle system.

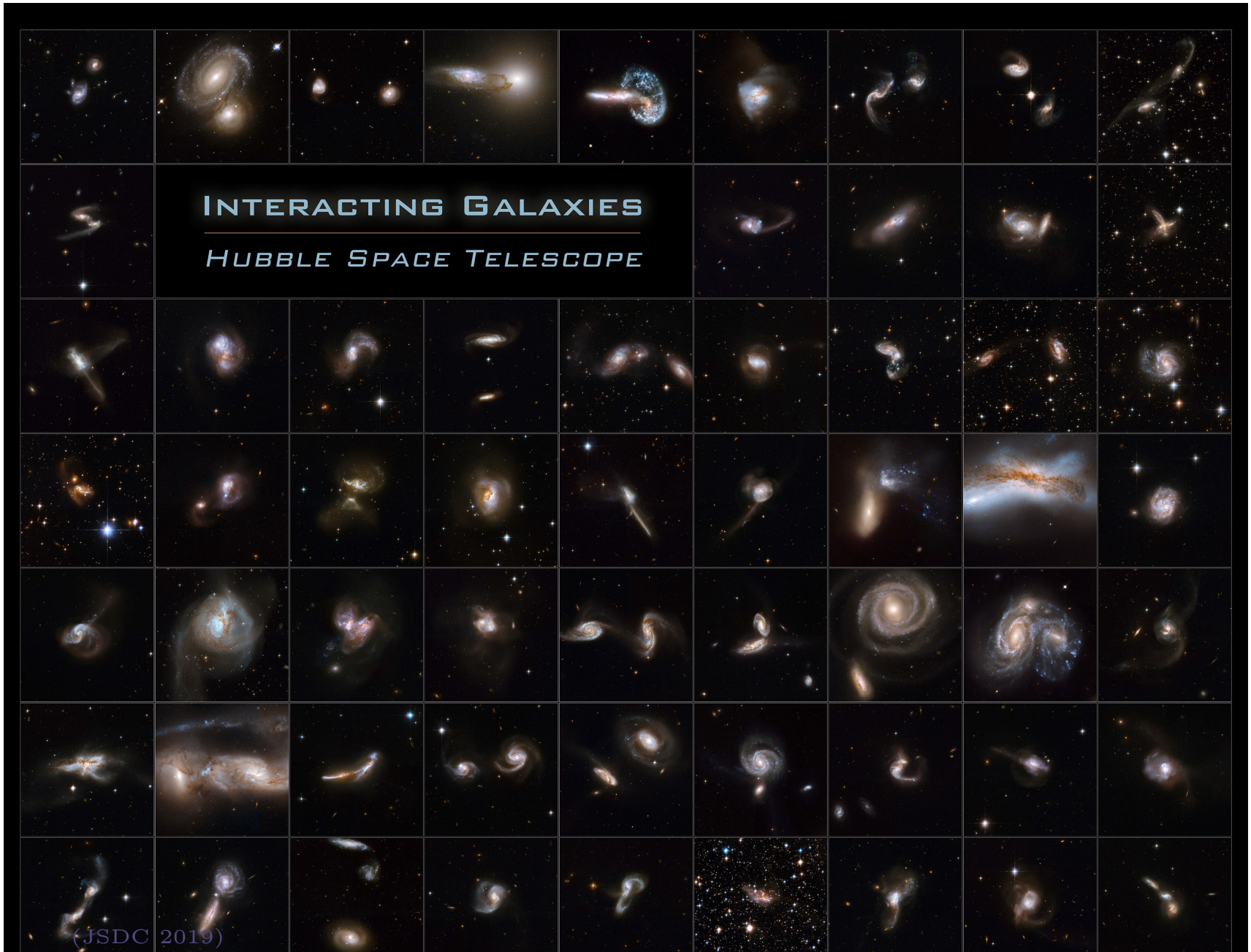


Credits: M. Jorba. PhD Thesis. (2019)

The BR4BP: yes and no

- The BR4BP is not a coherent model: the primaries does not verify Newton's laws. The Quasi-Bicircular Problem is a coherent version of the BR4BP: needs the computation of a quasi-bicircular solution of the three body problem.

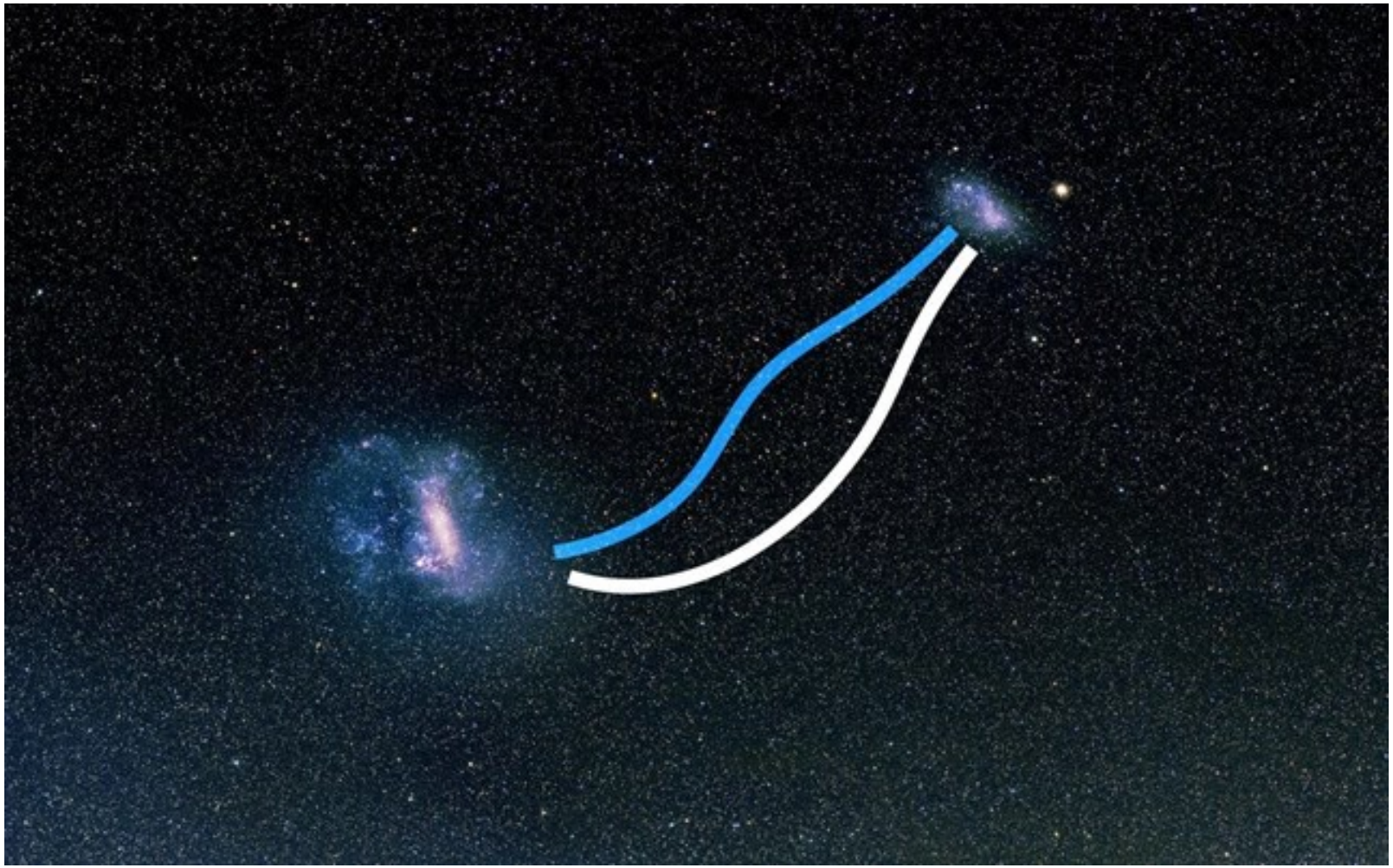
- The Solar Radiation Pressure can be included in both the BR4BP and the Quasi-Bicircular model



(JSDC 2019)

Galaxies NGC 3808A (right) and NGC 3808B (left).

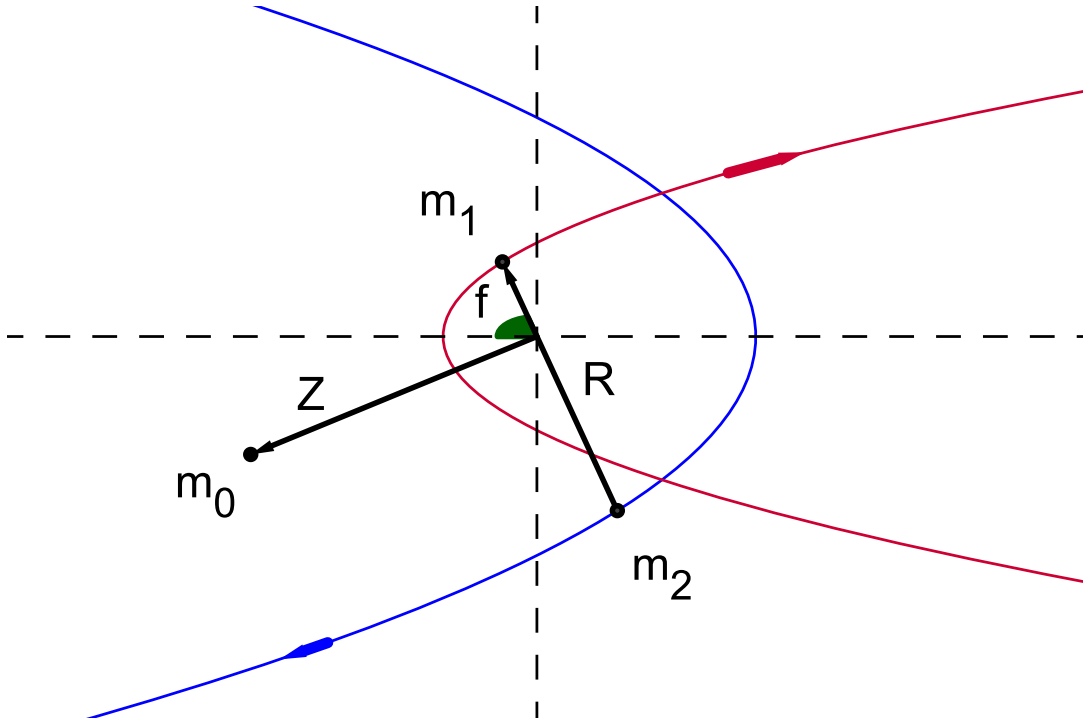
Credits: NASA, Hubble Space Telescope (2015)



Magellanic system

Credits: Belokurov et al. MNRAS (2017)

The Planar Parabolic Restricted Three-Body Problem



$$\frac{d^2\mathbf{Z}}{dt^2} = -(1-\mu)\frac{\mathbf{Z}-\mathbf{Z}_1}{|\mathbf{Z}-\mathbf{Z}_1|^3} - \mu\frac{\mathbf{Z}-\mathbf{Z}_2}{|\mathbf{Z}-\mathbf{Z}_2|^3},$$

The Planar Parabolic Restricted Three-Body Problem

Compatification to extend the flow when $s \to \pm\infty$: $\sin(\theta) = \tanh(s)$.

$$\begin{cases} \dfrac{d\theta}{ds} &= \cos\theta, \\ \dfrac{d\mathbf{z}}{ds} &= \mathbf{w}, \\ \dfrac{d\mathbf{w}}{ds} &= -A(\theta)\mathbf{w} + \nabla\Omega(\mathbf{z}) \end{cases} \qquad A(\theta) = \begin{pmatrix} \sin\theta & 4\cos\theta \\ -4\cos\theta & \sin\theta \end{pmatrix},$$

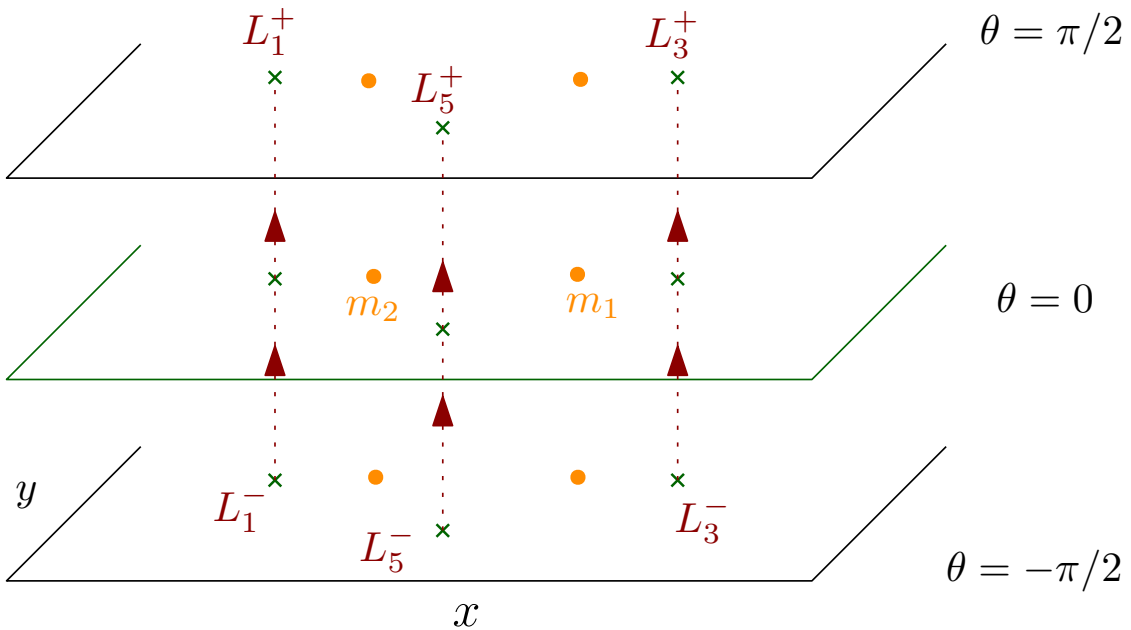
$$\Omega(\mathbf{z}) = x^2 + y^2 + 2\frac{1-\mu}{\sqrt{(x-\mu)^2+y^2}} + 2\frac{\mu}{\sqrt{(x-\mu+1)^2+y^2}}.$$

$\theta = \pm\pi/2$ are invariant $\quad\to\quad$ Boundary problems (infinity)

The Planar Parabolic Restricted Three-Body Problem

10 equilibrium points at the boundaries of the phase space that correspond to "infinity": $L^{\pm}_{1,2,3}$, collinear points, $L^{\pm}_{4,5}$ triangular points

	$L^+_{1,2,3}$	$L^+_{4,5}$
$\dim(W^u)$	1	2
$\dim(W^s)$	4	3



The $W^s(L_i^+)$ and $W^u(L_i^-)$, $i = 1, 2, 3$ are of co-dimension 1.

Generating bridges and tails

Dynamics forwards in time:

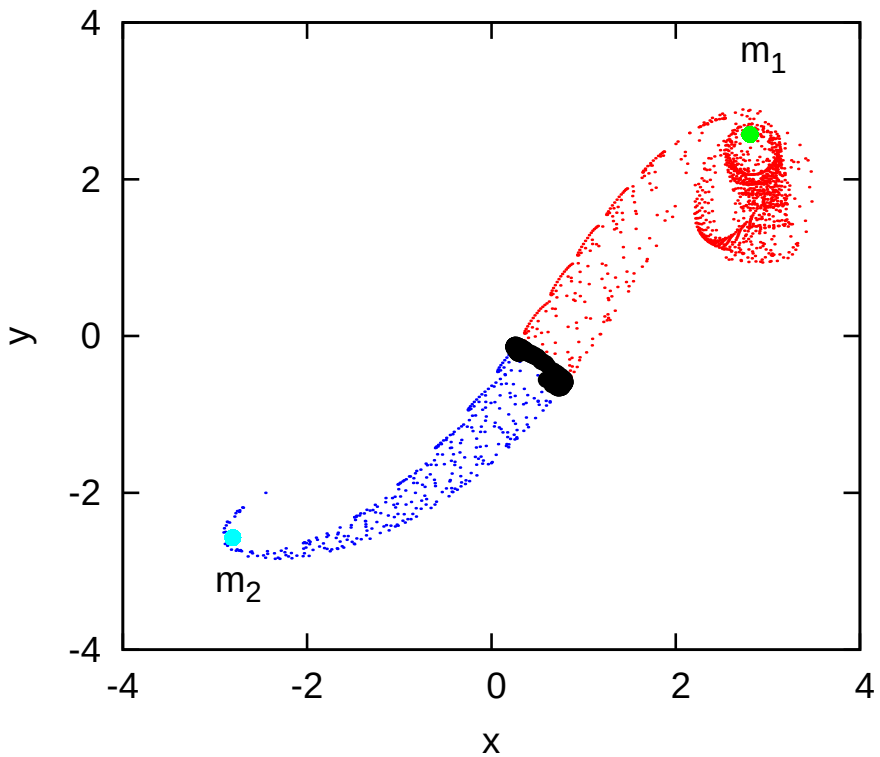
- There exist of only two different types of final evolutions: capture orbit (around a primary) or escape orbit (far away from both primaries). Collision manifolds: $W^{u/s}(m_j)$.

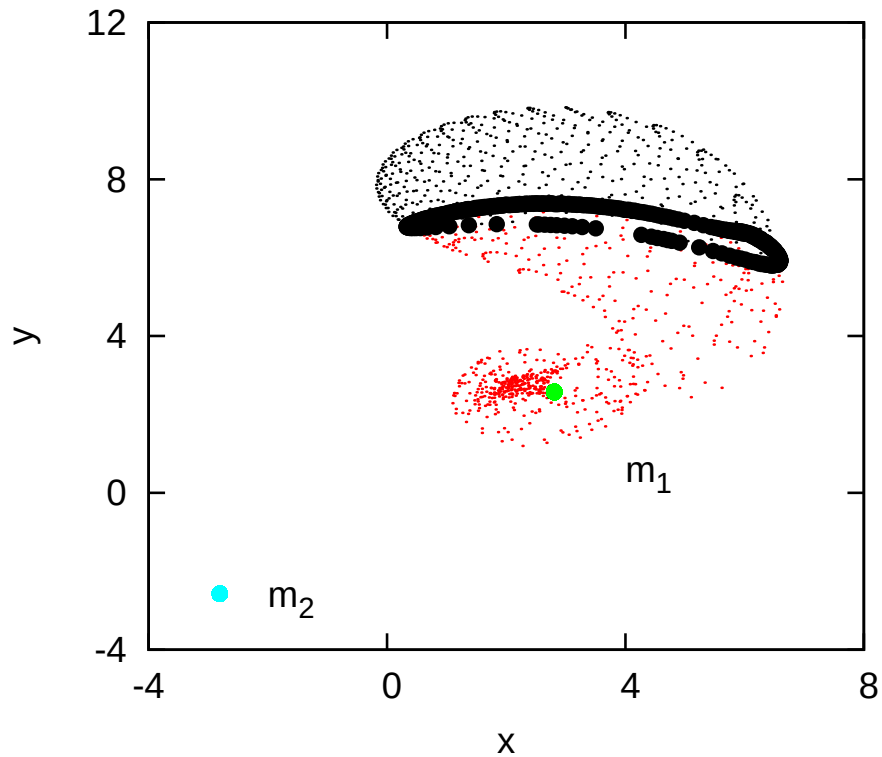
- The invariant manifolds $W^u(L^-_{1,2,3})$ and $W^s(L^+_{1,2,3})$ are of codimension 1: they behave as a frontier and divide the phase space in regions where only one of the two final evolutions are allowed: escape or capture.

- Heteroclinic connections between the equilibrium points and the collision manifolds:

$$\boxed{W^s(L_i^+) \cap W^u(m_j), \qquad i = 1, 2, 3 \quad j = 1, 2}$$

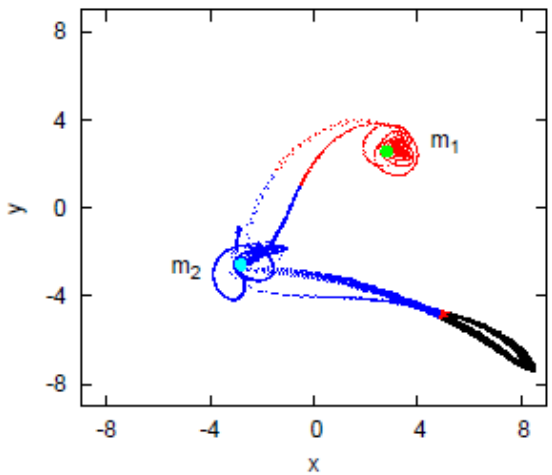
Bridges and Tails



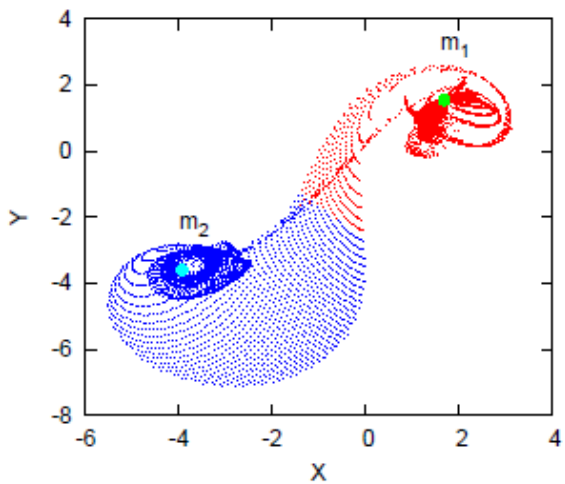


$$\mu = 0.5$$

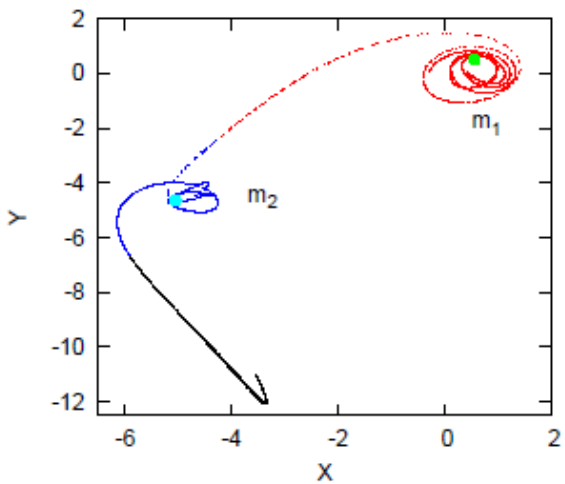
Bridges and Tails



$\mu = 0.5$



$\mu = 0.3$



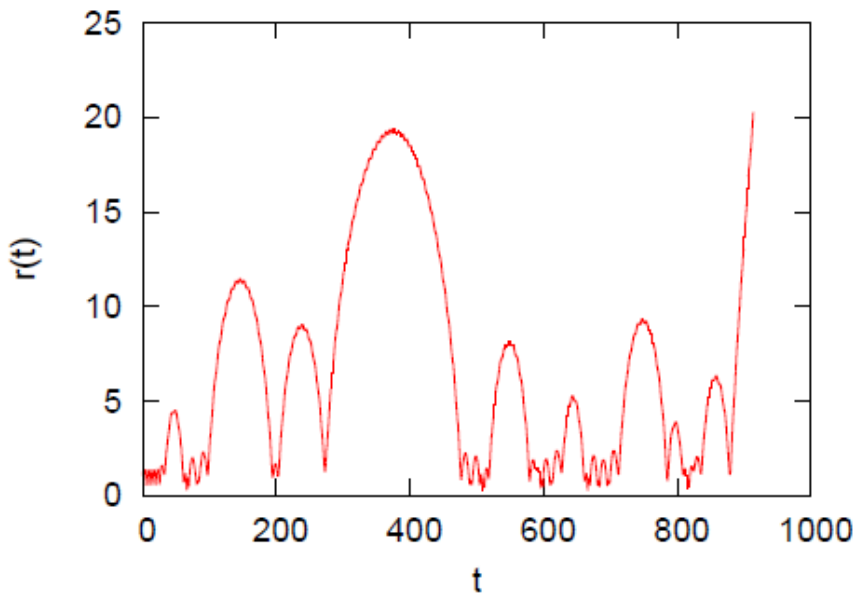
$\mu = 0.1$

Atomic dynamics

The same methodologies/ideas can be applied to atomic (classic) dynamics:

- ionization in a hydrogen atom in a circularly polarized microwave field

$$H = \frac{1}{2}(p_x^2 + p_y^2) - (xp_y - yp_x) - \frac{1}{r} + Kx$$

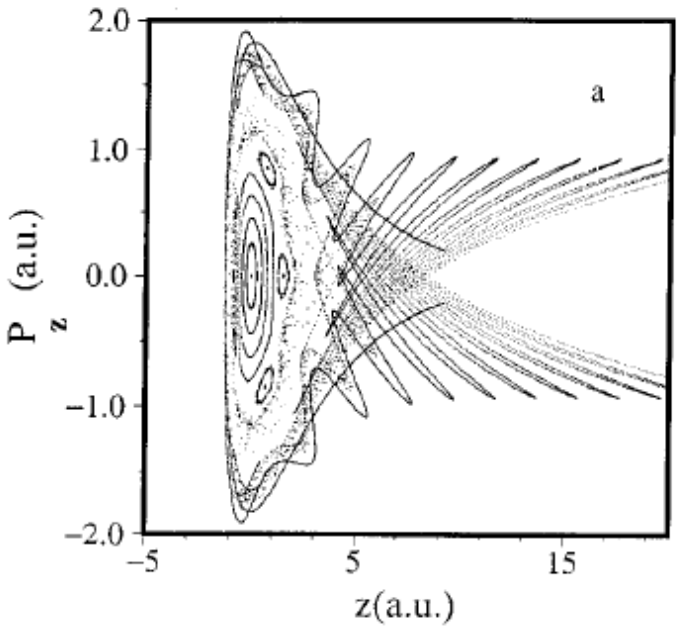


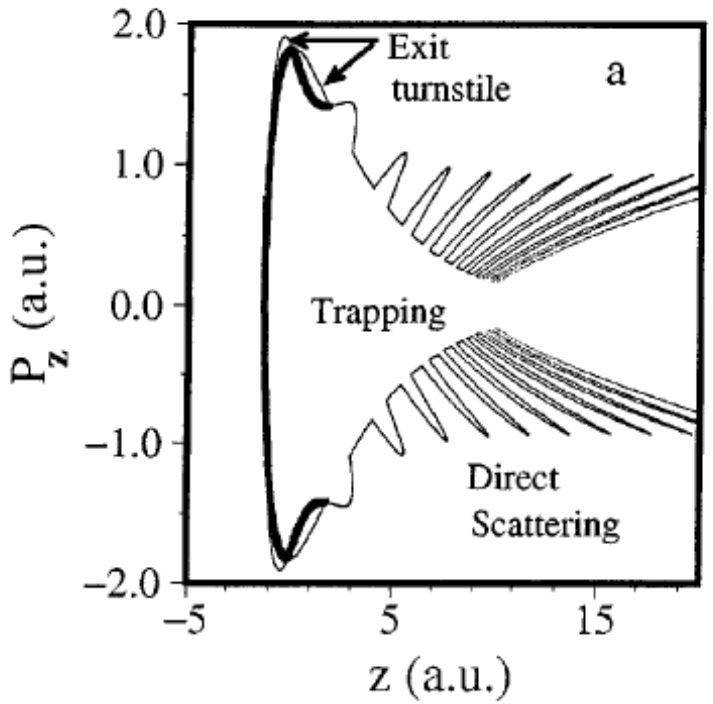
Atomic dynamics

The same methodologies/ideas can be applied to atomic (classic) dynamics:

- the scattering of the ^4He from different Cu surfaces

$$H = \frac{1}{K}(p_x^2 + p_z^2) + V(x, z)$$





Credits: Borondo, Guantes, Phys. Rev. E (1997)

Thank you for your attention