Unfolding of resonant saddles and the Dulac time

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Outline of the talk

PART 0 A motivating example
[D. Marín and J. Villadelprat, JDE 2013]

PART 1 Temporal normal forms
[P. Mardešić, D. Marín and J. Villadelprat, DCDS 2007]

PART 2 Asymptotic expansion of the Dulac time
[Work in progress with D. Marín]

A motivating example

Consider the family of polynomial planar vector fields

$$X_\mu(x,y) = y(x-1)\partial_x + (x+\mu y^2)\partial_y \text{ with } \mu \in (0, \tfrac{1}{2}).$$

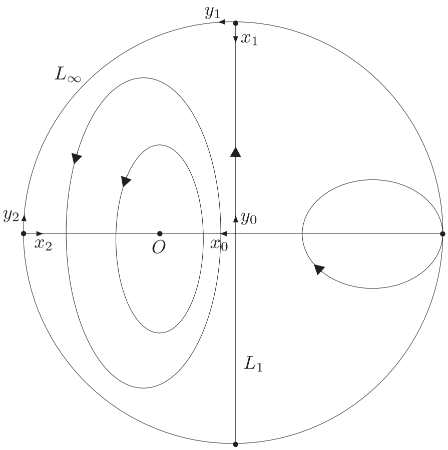
It has a center at the origin as unique critical point. Since its period annulus is unbounded, we compactify $\mathbb{R}^2$ to $\mathbb{RP}^2$. So we consider the coordinates of $\mathbb{RP}^2$ given by $(x_1, y_1) = \left(\frac{1}{y}, \frac{1-x}{y}\right)$ and $(x_2, y_2) = \left(\frac{1}{1-x}, \frac{y}{1-x}\right)$, which yield to

$$X_\mu(x_1,y_1) = \tfrac{1}{x_1}\big(x_1(-\mu - x_1^2 + x_1 y_1)\partial_{x_1} + y_1(1-\mu - x_1^2 + x_1 y_1)\partial_{y_1}\big)$$

and

$$X_\mu(x_2,y_2) = \tfrac{1}{x_2}\big(-x_2 y_2 \partial_{x_2} + (-x_2 + x_2^2 + (\mu-1)y_2^2)\partial_{y_2}\big).$$

A motivating example



$(x_1, y_1) = (0,0)$ is a hyperbolic saddle of $x_1 X_\mu$
$(x_2, y_2) = (0,0)$ is a degenerate singularity of $x_2 X_\mu$

A motivating example

We blow-up $(x_2, y_2) = (0,0)$ taking (t_1, x_2) and (s_1, y_2) with $y_2 = t_1 x_2$ and $x_2 = s_1 y_2$, which yield to

$$X_\mu(t_1, x_2) = \tfrac{1}{x_2}\big((-1 + x_2 + \mu t_1^2 x_2)\partial_{t_1} - t_1 x_2^2 \partial_{x_2}\big)$$

and

$$\begin{aligned} X_\mu(s_1, y_2) = \tfrac{1}{s_1 y_2}\big(&s_1(s_1 - \mu y_2 - s_1^2 y_2)\partial_{s_1} \\ &+ y_2(-s_1 + (\mu - 1)y_2 + s_1^2 y_2)\partial_{y_2}\big) \end{aligned}$$

respectively. Note that $x_2 X_\mu(t_1, x_2)$ has not any singularity along $x_2 = 0$. In the second chart, $s_1 y_2 X_\mu(s_1, y_2)$ still has a degenerate singularity at $(s_1, y_2) = (0,0)$ and so we must blow-up again.

A motivating example

To this end we take two new charts coordinated by means of (s_1, t_2) and (s_2, y_2) with $y_2 = t_2 s_1$ and $s_1 = s_2 y_2$. The expression of X_μ in these charts is

$$\begin{aligned} X_\mu(s_1, t_2) = \tfrac{1}{s_1 t_2}\big(&s_1(1 - \mu t_2 - s_1^2 t_2)\partial_{s_1} \\ &+ t_2(-2 + (2\mu - 1)t_2 + 2s_1^2 t_2)\partial_{t_2}\big) \end{aligned}$$

and

$$\begin{aligned} X_\mu(s_2, y_2) = \tfrac{1}{s_2 y_2}\big(&s_2(1 - 2\mu + 2s_2 - 2s_2^2 y_2^2)\partial_{s_2} \\ &+ y_2(\mu - 1 - s_2 + s_2^2 y_2^2)\partial_{y_2}\big) \end{aligned}$$

respectively, which have only hyperbolic saddles at the origin.

A motivating example

At this point we rename the new coordinates in order to unify the notation and we also give their expressions in terms of the original (x, y) coordinates:

$$(u_1, v_1) := (y_1, x_1) = \left(\frac{1-x}{y}, \frac{1}{y}\right) \qquad (u_3, v_3) := (s_1, t_2) = \left(\frac{1}{y}, \frac{y^2}{1-x}\right)$$

$$(u_2, v_2) := (s_2, y_2) = \left(\frac{1-x}{y^2}, \frac{y}{1-x}\right) \quad (u_4, v_4) := (t_1, x_2) = \left(y, \frac{1}{1-x}\right)$$

A motivating example

We obtain in addition the following vector fields

$$X_\mu(u_i,v_i)=\frac{1}{u_i^{m_i}v_i^{n_i}}\big(u_iP_i(u_i,v_i)\partial_{u_i}+v_iQ_i(u_i,v_i)\partial_{v_i}\big)$$

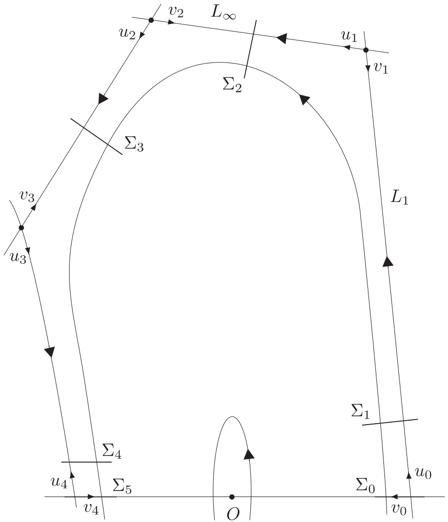
for $i=1,2,3$, with

$$\begin{aligned} &P_1(u,v)=1-\mu+uv-v^2 && (m_1,n_1)=(0,1)\\ &Q_1(u,v)=-\mu+uv-v^2 && \lambda_1=\tfrac{\mu}{1-\mu}\end{aligned}$$

$$\begin{aligned} &P_2(u,v)=1-2\mu+2u-2u^2v^2 && (m_2,n_2)=(1,1)\\ &Q_2(u,v)=\mu-1-u+u^2v^2 && \lambda_2=\tfrac{1-\mu}{1-2\mu}\end{aligned}$$

$$\begin{aligned} &P_3(u,v)=1-\mu v-u^2v && (m_3,n_3)=(1,1)\\ &Q_3(u,v)=-2+(2\mu-1)v+2u^2v && \lambda_3=2\end{aligned}$$

A motivating example



TEMPORAL NORMAL FORMS

Temporal normal form

Let us consider a $\mathscr{C}^\infty$ unfolding $\{X_\mu\}_{\mu\in U}$ of a hyperbolic saddle point at the origin. More precisely

$$X_\mu := xA(x,y;\mu)\partial_x + yB(x,y;\mu)\partial_y$$

where

- U is an open set of $\mathbb{R}^M$,
- A and B belong to $\mathscr{C}^\infty(V\times U)$ for some open set V containing the origin,
- $A(0,0;\mu)=1$ and $\lambda(\mu):=-B(0,0;\mu)>0$ for all $\mu\in U$.

We also consider the collinear family

$$Y_\mu = \frac{1}{v}\,X_\mu,\ \text{where } v := x^m y^n \text{ and } m,n\in\mathbb{Z}.$$

Temporal normal form

- Two vector fields Z and W are conjugate if there exists a change of coordinates Φ transforming Z to W, i.e., $\Phi^\star Z = W$, where

$$\big(\Phi^\star Z\big)(p) := \big(D\Phi\big)_p^{-1}\big(Z \circ \Phi(p)\big).$$

The vector fields Z and W are equivalent at a point p_0, if they are conjugate up to a nonzero multiple: $\Phi^\star Z = fW$ with $f(p_0) \neq 0$.

When dealing with families of vector fields Z_μ and W_μ then we ask

$$\Phi : V \times U \longrightarrow V \times U$$

to preserve the planes $\mu =$constant, i.e., $\Phi(x, y, \mu) = (\Phi_\mu(x, y), \mu)$.

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  to preserve the planes $\mu =$constant, i.e., $\Phi(x, y, \mu) = (\Phi_\mu(x, y), \mu)$.

- $f : \mathbb{R}^2 \times U \longrightarrow \mathbb{R}$ is N-flat wrt (x, y) if it is $\mathscr{C}^{N+1}$ and verifies the estimates

$$\max\big\{|\partial_I^i f(x, y, \mu)| : |I| = i\big\} \leqslant C\|(x, y)\|^{N-i},\ i = 0, 1, \dots, N,$$

  in some neighbourhood of $(0, 0, \mu_0) \in \mathbb{R}^2 \times U$ and a constant $C > 0$.

Temporal normal form

- Fix $\mu_0 \in U$ and set $\lambda_0 = \lambda(\mu_0)$. It is well-known that X_{μ_0} is $\mathscr{C}^\infty$ equivalent to

$$x\partial_x + \Big(-p/q + \sum_{i\geqslant 0} \alpha_{i+1}(x^p y^q)^i\Big) y\partial_y$$

if $\lambda_0 = p/q$ with $\gcd(p,q) = 1$. In case that $\lambda_0 \notin \mathbb{Q}$ then $\alpha_i = 0$ for all $i \in \mathbb{N}$.

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- The orbital codimension of X_{μ_0} is $\kappa = \infty$ if $\lambda_0 \notin \mathbb{Q}$ and, otherwise,

$$\kappa = \min\big\{i \in \mathbb{N} : \alpha_{i+1} \neq 0\big\}.$$

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$$\kappa = \min\big\{ i \in \mathbb{N} : \alpha_{i+1} \neq 0 \big\}.$$

- This is well defined because the monomial $(x^p y^q)^\kappa$ can not be annihilated by means of a smooth coordinate transformation preserving the normal form.

Temporal normal form

$$Y_\mu = \frac{1}{v}\left(xA(x,y;\mu)\partial_x + yB(x,y;\mu)\partial_y\right) \text{ with } v = x^m y^n.$$

Theorem A

Fix $\mu_0 \in U$ and set $\lambda_0 = \lambda(\mu_0)$. Then for any $k \in \mathbb{N}$ the family $\{Y_\mu\}_{\mu \in U}$ is $\mathscr{C}^k$ conjugate in a neighbourhood of $(0,0,\mu_0) \in \mathbb{R}^2 \times U$ to

$$Y_\mu^{NF} = \frac{1}{v + u^\ell Q_\mu(u)}\Big(x\partial_x + \big(-\lambda(\mu) + P_\mu(u)\big)y\partial_y\Big),$$

where

(a) if $\lambda_0 \notin \mathbb{Q}$ then $P_\mu \equiv Q_\mu \equiv 0$,

(b) if $\lambda_0 = p/q$ with $\gcd(p,q) = 1$ then P_μ and Q_μ are polynomials in the monomial $u = x^p y^q$ and $\ell = \min\big\{\beta \in \mathbb{Z} : \beta(p,q) \geqslant (m,n)\big\}$.

Moreover in case that X_{μ_0} has orbital codimension $\kappa < \infty$ then $\deg P_\mu \leqslant 2\kappa$ and $\deg Q_\mu \leqslant \kappa - \min(\ell, 1)$.

Preliminary results

$$X_\mu = xA(x,y;\mu)\partial_x + yB(x,y;\mu)\partial_y$$

[Roussarie 1975], [Samovol 1982], [Il'yashenko and Yakovenko 1991], ...

Theorem 1

Fix $\mu_0 \in U$ and set $\lambda_0 = \lambda(\mu_0)$. Then for any $k \in \mathbb{N}$ the family $\{X_\mu\}_{\mu \in U}$ is $\mathscr{C}^k$ equivalent in a neighbourhood of $(0,0,\mu_0) \in \mathbb{R}^2 \times U$ to

$$X_\mu^{NF} = x\partial_x + \big(-\lambda(\mu) + P_\mu(u)\big)y\partial_y,$$

where

(a) if $\lambda_0 \notin \mathbb{Q}$, then $P_\mu \equiv Q_\mu \equiv 0$,

(b) if $\lambda_0 = p/q$ with $\gcd(p,q) = 1$, then P_μ and Q_μ are polynomials in the resonant monomial $u = x^p y^q$.

Moreover, if X_{μ_0} has orbital codimension $\kappa < \infty$ then $\deg P_\mu \leqslant 2\kappa$.

Sketch of the proof of Theorem 1

- Let H^h be the vector space of homogeneous vector fields of degree h and let $L = x\partial_x - \lambda(\mu)y\partial_y \in H^1$ be the linear part of X_μ. Then

$$[L, x^i y^j \partial_x] = \big(1 - i + j\lambda(\mu)\big)x^i y^j \partial_x$$
$$[L, x^i y^j \partial_y] = \big(-i + (j-1)\lambda(\mu)\big)x^i y^j \partial_y$$

For each h, the mapping $[L, \cdot\,] : H^h \longrightarrow H^h$ is linear and

$$H^h = \big(\mathrm{Im}[L, \cdot\,]\big) \oplus \big(\mathrm{Ker}[L, \cdot\,]\big).$$

- For any N there exists a polynomial change of coordinates transforming the vector field family X_μ to the form

$$x\partial_x - \lambda(\mu)y\partial_y + g_2 + \cdots + g_N + R(x,y),$$

where $g_h \in \mathrm{Ker}[L, \cdot\,]$ for $h = 1, \ldots, N$ and $R(x,y) = \mathrm{o}(\|(x,y)\|^N)$.

Sketch of the proof of Theorem 1

- If $\lambda_0 \notin \mathbb{Q}$ then, for $\mu \approx \mu_0$, X_μ is linearizable up to an N-flat term for any N.

  If $\lambda_0 = p/q$ with $\gcd(p,q) = 1$ then, for $\mu \approx \mu_0$ and up to an N-flat term, all monomials can be eliminated except for the resonant ones:

  $$u^k x\partial_x \text{ and } u^k y\partial_y \text{ with } u = x^p y^q.$$

  When working with equivalence and not conjugacy relation, it is legitimate to divide by the component of $x\partial_x$, so that we get $X^{NF} + \hat{R}$ with

  $$X^{NF} := x\partial_x + \big(-\lambda(\mu) + P_\mu(u)\big) y\partial_y \text{ and } \hat{R}(x,y) = \mathrm{o}(\|(x,y)\|^N)$$

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- Homotopic method

  The vector fields F and $F + w$ are $\mathscr{C}^k$ conjugate if the homological equation

  $$[F + \tau w, Z_\tau] = w$$

  has a $\mathscr{C}^k$ solution Z_τ.

Sketch of the proof of Theorem 1

- Existence and regularity

  There exists $N = N(k,F)$ such that if w is N-flat then the homological equation $[F + \tau w, Z_\tau] = w$ has a a $\mathscr{C}^k$ solution Z_τ.

Sketch of the proof of Theorem 1

- Existence and regularity

  There exists $N = N(k, F)$ such that if w is N-flat then the homological equation $[F + \tau w, Z_\tau] = w$ has a a $\mathscr{C}^k$ solution Z_τ.

- Delicate point

  How is the dependence of $N(k, X^{NF})$ with respect to the original X?

  Roussarie works with $k = N = \infty$ and his proof applies when k is finite but in that case $N(x, X)$ depends on $\|X\|$. This is a problem if the norm of X grows along the process of annihilation of non-resonant monomials.

  Il'yashenko-Yakovenko do not pay too much attention to this point.

  Samovol shows that $N(k, F)$ depends on k and the linear part of F, which remains fixed along all the process.

Preliminary results

Lemma (Teyssier 2004)

Let $\varphi_\mu(t;x,y)$ be the flow of X_μ and consider any function F with $F(0,0)=0$. Then $\Phi_\mu(x,y) := \varphi_\mu(F(x,y);x,y)$ is a family of local diffeomorphisms with $\Phi_\mu(0,0)=0$ such that

$$(\Phi_\mu)^\star(X_\mu) = \frac{1}{1+X_\mu(F)} X_\mu$$

Preliminary results

Recall that $Y_\mu = \frac{1}{v} X_\mu$ where X_μ is a $\mathscr{C}^\infty$ unfolding of a saddle.

Lemma 2

Given a function f with $f(0,0) = 0$ there exists a family of local diffeomorphisms Φ_μ with $\Phi_\mu(0,0) = 0$ such that

$$(\Phi_\mu)^\star(Y_\mu) = \frac{1}{v + X_\mu(vf)} X_\mu \text{ on } xy \neq 0.$$

In fact $\Phi_\mu(x,y) := \varphi_\mu(F(x,y); x, y)$ where F is defined implicitly by

$$vf(x,y) = \int_0^{F(x,y)} v \circ \varphi_\mu(t; x, y) dt.$$

Notation: $\Phi_\mu = \Phi[Y_\mu, f_\mu]$

Preliminary results

Theorem 3

Fix $\mu_0 \in U$ and set $\lambda_0 = \lambda(\mu_0)$. For any $k \in \mathbb{N}$ there exists $N = N\big(k, \lambda_0, m, n\big)$ such that if $\{h_\mu\}$ is a $\mathscr{C}^N$ family of N-flat functions then the homological equation

$$X_\mu(v f_\mu) = v h_\mu$$

has a $\mathscr{C}^k$ family of solutions $\{f_\mu\}$ defined in a neighbourhood of $(0, 0, \mu_0) \in \mathbb{R}^2 \times U$.

Preliminary results

Theorem 3

Fix $\mu_0 \in U$ and set $\lambda_0 = \lambda(\mu_0)$. For any $k \in \mathbb{N}$ there exists $N = N\big(k, \lambda_0, m, n\big)$ such that if $\{h_\mu\}$ is a $\mathscr{C}^N$ family of N-flat functions then the homological equation

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has a $\mathscr{C}^k$ family of solutions $\{f_\mu\}$ defined in a neighbourhood of $(0, 0, \mu_0) \in \mathbb{R}^2 \times U$. More precisely, we can take $N(k, \lambda_0, m, n)$ as

$$2\left[\max\{(\nu_0 + 1)k - m + \lambda_0 n, (\nu_0/\lambda_0 + 1)k + m/\lambda_0 - n\} + 1\right],$$

where $\nu_0 = \max\{1, \lambda_0\}$ and $[\,\cdot\,]$ denotes the integer part.

Sketch of the proof of Theorem 3

- ANSATZ FOR $X(F) = H$

  Let φ_t be the flow of X. If

$$F(x,y) = \int_{\pm\infty}^{0} H \circ \varphi(t;x,y)dt$$

is a well-defined smooth function then it is a solution of the homological equation $X(F) = H$. Indeed

$$X(F) = \frac{d}{ds}\int_{\pm\infty}^{0} H \circ \varphi_t \circ \varphi_s dt\bigg|_{s=0} = \frac{d}{ds}\int_{\pm\infty}^{s} H \circ \varphi_\tau d\tau\bigg|_{s=0} = H$$

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- We want to find f such that $X(vf)=vh$ with $v=x^m y^n$, $m,n\in\mathbb{Z}$, and a given N-flat function h.

  Without loss of generality we can consider $X^{NF}=x\partial_x+(-\lambda(\mu)+P_\mu(u))y\partial_y$ instead of X.

Sketch of the proof of Theorem 3

- GLOBALIZATION

  Take a $\mathscr{C}^\infty$ bump function $\psi_\varepsilon : \mathbb{R}^2 \longrightarrow [0,1]$ such that $\psi_\varepsilon(p) = 1$ if $\|p\| \leqslant \varepsilon/2$, $\psi_\varepsilon(p) = 0$ if $\|p\| \geqslant \varepsilon$ and $\|D\psi_\varepsilon\| < c/\varepsilon$.

  Define $X_\varepsilon := x\partial_x + (-\lambda_0 + P_\varepsilon(x,y))y\partial_y$ where $P_\varepsilon := (\lambda_0 - \lambda + P)\psi_\varepsilon$.

  X_ε coincides with X^{NF} on $D_{\varepsilon/2}(0)$ and is linear outside $D_\varepsilon(0)$. So its flow φ_t^ε is globally defined.

  Replace h by the global function $h\psi_\varepsilon$, which is also N-flat, and consider the homological equation $X_\varepsilon(vf_\varepsilon) = vh_\varepsilon$.

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  Replace h by the global function $h\psi_\varepsilon$, which is also N-flat, and consider the homological equation $X_\varepsilon(vf_\varepsilon) = vh_\varepsilon$.

- DECOMPOSITION OF THE DISCREPANCY

  Since h_ε is N-flat, setting $M = [N/2]$, we can write $h_\varepsilon = h_1 + h_2$ with h_1 M-flat with respect to x and h_2 M-flat with respect to y.

Sketch of the proof of Theorem 3

- Thus the homological equation writes as $X_\varepsilon(vf_\varepsilon) = vh_1 + vh_2$, which lead us to choose f_ε such that

$$vf_\varepsilon(x,y) = \int_{-\infty}^{0} (vh_1)\circ\varphi_\varepsilon(t;x,y)dt + \int_{+\infty}^{0} (vh_2)\circ\varphi_\varepsilon(t;x,y)dt,$$

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  where φ_ε is the (complete) flow of X_ε.

- $\frac{d}{dt}(v\circ\varphi_\varepsilon) = (v\circ\varphi_\varepsilon)(m-\lambda_0 n + nP_\varepsilon(\varphi_\varepsilon))$, that yields

$$\frac{v\circ\varphi_\varepsilon}{v} = e^{(m-\lambda_0 n)t}\exp\left(n\int_0^t P_\varepsilon(\varphi_\varepsilon(s))ds\right)$$

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- Thus $f_\varepsilon(x,y,\mu) = \int_{-\infty}^{0} \mathcal{I}_1^\varepsilon(x,y,\mu,t)dt + \int_{+\infty}^{0} \mathcal{I}_2^\varepsilon(x,y,\mu,t)dt$ with

$$\mathcal{I}_i^\varepsilon(p,t,\mu) := e^{(m-\lambda_0 n)t}(h_i\circ\varphi_\varepsilon(t;p,\mu))\exp\left(n\int_0^t P_\varepsilon(\varphi_\varepsilon(s;p,\mu))ds\right)$$

Sketch of the proof of Theorem 3

Next we bound the derivatives of $\mathcal{I}_i^{\varepsilon}(x, y, \mu, t)$ with respect to x, y and μ using

- Multivariate version of the chain rule with higher derivates
- Grönwall's Lemma
- Flatness properties of h_i

to show that, for $\mu \approx \mu_0$ and $\varepsilon \approx 0$, $\partial_I^i \mathcal{I}_1^{\varepsilon}$ and $\partial_I^i \mathcal{I}_2^{\varepsilon}$ are integrable with respect to t on $(-\infty, 0)$ and $(0, +\infty)$ respectively. Then we conclude by applying the Dominated Convergence Theorem.

Setting $V_\delta = \mathbb{R}^2 \times \{\|\mu - \mu_0\| < \delta\}$, we have the estimates $\|P_\mu^\varepsilon\|_{V_\delta} \leqslant \eta(\varepsilon,\delta)$ and $\|DX_\mu^\varepsilon\|_{V_\delta} \leqslant \nu(\varepsilon,\delta)$ where η and ν are continuous functions with $\eta(0,0) = 0$ and $\nu(0,0) = \nu_0 = \max\{1,\lambda_0\}$. Then, for $0 \leqslant i \leqslant k$,

$$|\partial_i^I \mathcal{I}_1^\varepsilon(x,y,\mu,t)| \leqslant K|x|^{M-k}e^{\alpha_1 t}$$
$$\text{with } \alpha_1 := M - (\nu+1)k - |n|\eta + m - \lambda(\mu)n$$

$$|\partial_i^I \mathcal{I}_2^\varepsilon(x,y,\mu,t)| \leqslant K|y|^{M-k}e^{\alpha_2 t}$$
$$\text{with } \alpha_2 := -\lambda_0 M + (\eta+\lambda_0)k + (|n| - M + k)\eta + m - \lambda(\mu)n$$

Thanks to $N \geqslant N(k,\lambda_0,m,n)$, we get $\alpha_1 > 0$ and $\alpha_2 < 0$ for $(\varepsilon,\delta) \approx (0,0)$. ∎

Temporal normal form

Theorem A

Fix $\mu_0 \in U$ and set $\lambda_0 = \lambda(\mu_0)$. Then for any $k \in \mathbb{N}$ the family $\{Y_\mu\}_{\mu \in U}$ is $\mathscr{C}^k$ conjugate in a neighbourhood of $(0, 0, \mu_0) \in \mathbb{R}^2 \times U$ to

$$Y_\mu^{NF} = \frac{1}{v + u^\ell Q_\mu(u)} \Big(x\partial_x + \big(-\lambda(\mu) + P_\mu(u)\big) y\partial_y \Big),$$

where

(a) if $\lambda_0 \notin \mathbb{Q}$ then $P_\mu \equiv Q_\mu \equiv 0$,

(b) if $\lambda_0 = p/q$ with $\gcd(p, q) = 1$ then P_μ and Q_μ are polynomials in the monomial $u = x^p y^q$ and $\ell = \min\{\beta \in \mathbb{Z} : \beta(p, q) \geqslant (m, n)\}$.

Moreover in case that X_{μ_0} has orbital codimension $\kappa < \infty$ then $\deg P_\mu \leqslant 2\kappa$ and $\deg Q_\mu \leqslant \kappa - \min(\ell, 1)$.

Sketch of the proof of Theorem A

- Fix $k \in \mathbb{N}$ and $\mu_0 \in U$, and let $N = (k, \lambda_0, m, m)$ be the integer given by Theorem 1. Take $s > N$. Then there exists a $\mathscr{C}^s$ diffeomorphism Φ_μ^0 such that

$$Y_\mu^1 := \left(\Phi_\mu^0\right)^\star (Y_\mu) = \frac{1}{v} \frac{X_\mu^{NF}}{1 + R_\mu(x,y)},$$

where $X_\mu^{NF} = x\partial_x + \big(-\lambda(\mu) + P_\mu(u)\big) y\partial_y$ and $R_\mu(0,0) = 0$.

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- By Lemma 2, if $\Phi_\mu^1 := \Phi[Y_\mu^1, f_\mu^1]$ then

$$\left(\Phi_\mu^1\right)^\star (Y_\mu^1) = \frac{X_\mu^{NF}}{v\big(1 + R_\mu(x,y)\big) + X_\mu^{NF}(v f_\mu^1)}.$$

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$$Y^1_\mu := \left(\Phi^0_\mu\right)^\star (Y_\mu) = \frac{1}{v} \frac{X^{NF}_\mu}{1 + R_\mu(x,y)},$$

where $X^{NF}_\mu = x\partial_x + \big(-\lambda(\mu) + P_\mu(u)\big) y\partial_y$ and $R_\mu(0,0) = 0$.

- By Lemma 2, if $\Phi^1_\mu := \Phi[Y^1_\mu, f^1_\mu]$ then

$$\left(\Phi^1_\mu\right)^\star (Y^1_\mu) = \frac{X^{NF}_\mu}{v\big(1 + R_\mu(x,y)\big) + X^{NF}_\mu(v f^1_\mu)}.$$

- Since $X^{NF}_{\mu_0}(x^a y^b) = \big(a - \lambda_0 b\big) x^a y^b + b P_{\mu_0}(u) x^a y^b$, $u^\ell \mathbb{R}[u]$ is a supplementary subspace of the image of $X^{NF}_{\mu_0}$ in $v\mathbb{R}[x,y]$.

Sketch of the proof of Theorem A

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- Take $f_\mu^1 \in \mathbb{R}[x,y]$ such that $vR_\mu(x,y) + X_\mu^{NF}(v f_\mu^1) = u^\ell Q_\mu(u) - v h_\mu(x,y)$ for some polynomial $Q_\mu(u)$ and some s-flat function $h_\mu(x,y)$.

Sketch of the proof of Theorem A

- Thus
$$Y_\mu^2 := \left(\Phi_\mu^1\right)^\star (Y_\mu^1) = \frac{X_\mu^{NF}}{v + u^\ell Q_\mu(u) - v h_\mu(x,y)}$$

Sketch of the proof of Theorem A

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- By Theorem 3, thanks to $s > N$, there exists a $\mathscr{C}^k$ function f_μ^2 such that $X_\mu^{NF}(v f_\mu^2) = v h_\mu$.

Sketch of the proof of Theorem A

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- We take $\Phi_\mu^2 := \Phi[Y_\mu^2, f_\mu^2]$ and then by Lemma 2

$$Y_\mu^3 := \big(\Phi_\mu^2\big)^\star(Y_\mu^2) = \frac{X_\mu^{NF}}{v + u^\ell Q_\mu(u) - v h_\mu(x,y) + X_\mu^{NF}(v f_\mu^2)} = \frac{X_\mu^{NF}}{v + u^\ell Q_\mu(u)}$$

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- Assume now $\kappa = \text{codim}\, X_{\mu_0} = \text{ord}_{u=0} P_{\mu_0} < \infty$. We take $\Phi_\mu^3 := \Phi[Y_\mu^3, f_\mu^3]$ with f_μ^3 so that
$$\left(\Phi_\mu^3\right)^\star (Y_\mu^3) = \frac{X_\mu^{NF}}{v + u^\ell Q_\mu(u) + X_\mu^{NF}(v f_\mu^3)}$$

Sketch of the proof of Theorem A

- Observe that if $u^\ell | \tau_\mu(u)$ then $f_\mu^3(x,y) = \tau_\mu(u)/v$ is regular at the origin and $X_\mu^{NF}(v f_\mu^3) = \tau_\mu'(u) u (p - \lambda(\mu) q + P_\mu(u))$.

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- By the Weierstrass Preparation Theorem there exists $B_\mu\in\mathbb{R}[u]$ of degree $\leqslant\kappa$ such that $p-\lambda(\mu)q+P_\mu(u)=A_\mu(u)B_\mu(u)$ with $A_{\mu_0}(0)\neq 0$. Thus

$$\left(\Phi_\mu^3\right)^\star(Y_\mu^3)=\frac{X_\mu^{NF}}{v+u^\ell Q_\mu(u)+u\tau'(u)A_\mu(u)B_\mu(u)}$$

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- If $u^\ell Q_\mu(u) = \sum_{i=\ell}^{r} a_i(\mu)u^i$, set $S_1(u) = \sum_{i=\ell}^{-1} a_i u^i$ and $S_2 = \sum_{i=0}^{r} a_i u^i$, so that

$$u^\ell Q_\mu + u\tau' A_\mu B_\mu = S_1 + S_2 + u\tau' A_\mu B_\mu$$

Sketch of the proof of Theorem A

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$$u^\ell Q_\mu + u\tau' A_\mu B_\mu = S_1 + S_2 + u\tau' A_\mu B_\mu$$

- Put $\nu = \max\{\ell, 1\}$ and consider the polynomial division $S_2 = u^\nu B_\mu C_\mu + R_\mu$ with $\deg R \leqslant \nu + \kappa - 1$. Hence

$$u^\ell Q_\mu + u\tau' A_\mu B_\mu = S_1 + u^\nu B_\mu C_\mu + R_\mu + u\tau' A_\mu B_\mu$$

Sketch of the proof of Theorem A

- The equality

$$u^\ell Q_\mu + u\tau' A_\mu B_\mu = S_1 + u^\nu B_\mu C_\mu + R_\mu + u\tau' A_\mu B_\mu$$

leads us to define

$$\tau_\mu(u) = -\int_0^u \xi^{\nu-1}\frac{C_\mu(\xi)}{A_\mu(\xi)}d\xi,$$

which is a smooth function at $(u,\mu)\approx(0,\mu_0)$ because $A_{\mu_0}(0)\neq 0$ and $\nu\geqslant 1$. Moreover it verifies $u^\ell|\tau_\mu(u)$ as desired due to $\nu\geqslant\ell$.

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- We get

$$\left(\Phi_\mu^3\right)^\star(Y_\mu^3) = \frac{X_\mu^{NF}}{v+S_1(u)+R(u)}$$

and by construction $S_1(u)+R(u)=u^\ell Q_\mu(u)$ with a polynomial Q_μ of degree $\kappa-\min(\ell,1)$.

■

ASYMPTOTIC EXPANSION OF THE DULAC TIME

The Dulac time

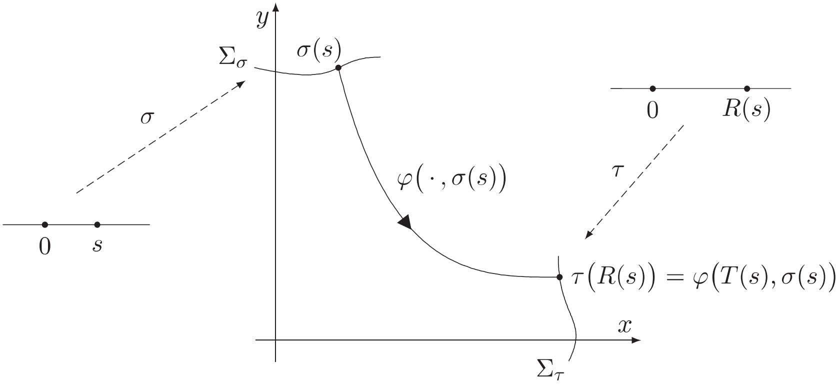
We consider a $\mathscr{C}^\infty$ unfolding of a saddle point at the origin with poles along one of its separatrices. More precisely, setting $\hat{\mu} := (\lambda, \mu) \in \hat{W} := (0, +\infty) \times W$ with W an open set of $\mathbb{R}^N$, let us take the family of vector fields $\{Y_{\hat{\mu}}\}_{\hat{\mu} \in \hat{W}}$ with

$$Y_{\hat{\mu}}(x,y) := \frac{1}{y^n}\Big(xP(x,y;\hat{\mu})\partial_x + yQ(x,y;\hat{\mu})\partial_y\Big),$$

where

- $n \in \mathbb{Z}_+ := \mathbb{N} \cup \{0\}$,
- P and Q belong to $\mathscr{C}^\infty(V \times \hat{W})$ for some open set V of $\mathbb{R}^2$ containing the origin,
- $P(x,0;\hat{\mu}) > 0$ and $Q(0,y;\hat{\mu}) < 0$ for all $(x,0),(0,y) \in V$ and $\hat{\mu} \in \hat{W}$,
- $\lambda = -\frac{Q(0,0;\hat{\mu})}{P(0,0;\hat{\mu})}$.

The Dulac time



The Dulac time $T(\,\cdot\,;\hat{\mu})$ between the transverse sections Σ_σ and Σ_τ.

The Dulac time

- The function defined for $s > 0$ and $\alpha \in \mathbb{R}$ by means of

$$\omega(s;\alpha) = \begin{cases} \frac{s^{-\alpha}-1}{\alpha} & \text{if } \alpha \neq 0, \\ -\log s & \text{if } \alpha = 0, \end{cases}$$

is called the Ecalle-Roussarie compensator.

The Dulac time

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  is called the Ecalle-Roussarie compensator.

- We denote by $\mathcal{I}_K(U)$ the set of $\mathscr{C}^K$ functions $h(s;\hat{\mu})$ defined on $(0,\varepsilon)\times U$, for some $\varepsilon > 0$, such that

$$\lim_{s\to 0^+} \mathscr{D}^j h(s;\hat{\mu}) = 0,$$

  uniformly on compacts sets of U, for all $j = 0, 1, \dots, K$ where $\mathscr{D} := s\partial_s$.
  We write $f \in \mathcal{I}_\infty(U)$ if $f \in \mathcal{I}_K(U)$ for all $K \in \mathbb{Z}_+$.

The Dulac time

- We say that $\{Y_{\hat{\mu}}\}_{\hat{\mu}\in\hat{W}}$ verifies the family linearization property (FLP) if there exist an open set $U \subset \mathbb{R}^2$ containing the origin and a $\mathscr{C}^\infty$ diffeomorphism $\Phi\colon U\times\hat{W} \longrightarrow V\times\hat{W}$ of the form $\Phi(x,y;\hat{\mu}) = (x + \text{h.o.t}, y + \text{h.o.t}; \hat{\mu})$ that, for each $\hat{\mu}$, conjugates $Y_{\hat{\mu}}$ with

$$\frac{1}{f(x,y;\hat{\mu})}(x\partial_x - \lambda y\partial_y)$$

where $f \in \mathscr{C}^\infty(U\times\hat{W})$.

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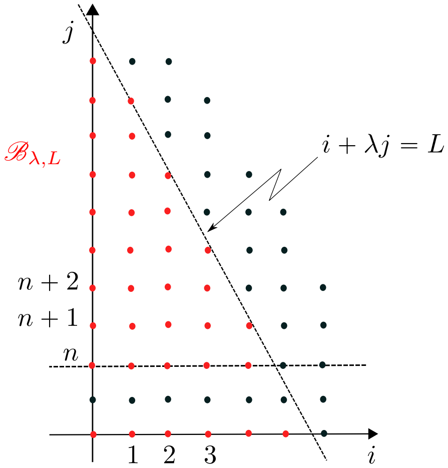
$$\frac{1}{f(x,y;\hat{\mu})}(x\partial_x - \lambda y\partial_y)$$

  where $f \in \mathscr{C}^\infty(U\times\hat{W})$.

- In what follows we denote $\Lambda_n := \mathbb{Z}_+ \times (\{0\}\cup\mathbb{Z}_{\geqslant n})$ and, for any λ and L positive real numbers, we define

$$\mathscr{B}_{\lambda,L} := \big\{(i,j)\in\Lambda_n : i + \lambda j \leqslant L\big\}.$$

The Dulac time



The Dulac time

Theorem B

Assume that $\{Y_{\hat{\mu}}\}_{\hat{\mu}\in\hat{W}}$ verifies the FLP. Let $T(\,\cdot\,;\hat{\mu})$ be the Dulac time between the transverse sections Σ_σ and Σ_τ and fix $\lambda_0 > 0$. For each $(i,j)\in\Lambda_n$ there exist a neighbourhood $I_{ij}^{\lambda_0}$ of λ_0 and a polynomial $P_{ij}^{\lambda_0}(z;\hat{\mu})\in\mathscr{C}^\infty\big(I_{ij}^{\lambda_0}\times W\big)[z]$ satisfying the following properties:

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(a) If $\lambda_0\notin\mathbb{Q}$ then $\deg_z P_{ij}^{\lambda_0}(z;\hat{\mu}) = 0$ and, otherwise, if $\lambda_0 = p/q$ with $\gcd(p,q)=1$, then

- (1) $\deg_z P_{ij}^{\lambda_0}(z;\hat{\mu}) \leqslant i/p$,
- (2) $\deg_z P_{ij}^{\lambda_0}(z;\hat{\mu}) = 0$ if $j=0$ and $iq<np$,
- (3) $P_{ij}^{\lambda_0}(z;\hat{\mu})\equiv 0$ if there exists $r\in\mathbb{N}$ such that $(i+rp, j-rq)\in\Lambda_n$.

Theorem B

(b) For each $L > 0$ there exists a neighbourhood $I_L^{\lambda_0}$ of λ_0 such that, for $s > 0$ small enough and $\hat{\mu} \in I_L^{\lambda_0} \times W$,

$$T(s;\hat{\mu}) = \Delta_0(\lambda)\log s + \sum_{(i,j)\in\mathscr{B}_{\lambda_0,L}} P_{ij}^{\lambda_0}\big(\omega(s;\alpha);\hat{\mu}\big)s^{i+\lambda j} + R(s;\hat{\mu}),$$

where $\Delta_0(\lambda) = 0$ if $n \in \mathbb{N}$ and $\Delta_0(\lambda) = -1/\lambda$ if $n = 0$, and

$$\alpha = \begin{cases} 0 & \text{if } \lambda_0 \notin \mathbb{Q}, \\ p - \lambda q & \text{if } \lambda_0 = p/q \text{ with } \gcd(p,q) = 1, \end{cases}$$

The Dulac time

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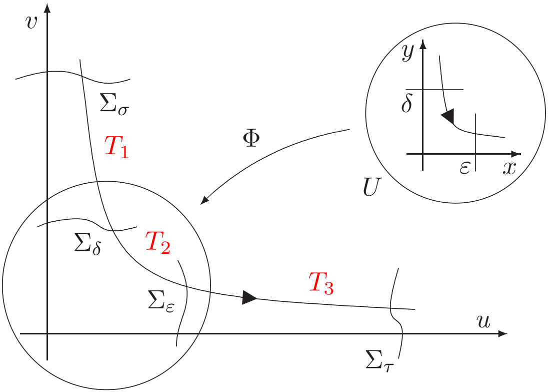
$$\alpha = \begin{cases} 0 & \text{if } \lambda_0 \notin \mathbb{Q}, \\ p - \lambda q & \text{if } \lambda_0 = p/q \text{ with } \gcd(p,q) = 1, \end{cases}$$

and the remainder writes as $R(s;\hat{\mu}) = s^L\Psi(s;\hat{\mu})$ with $\Psi \in \mathcal{I}_\infty\big(I_L^{\lambda_0} \times W\big)$. Moreover, in a neighbourhood of $\{0\} \times I_L^{\lambda_0} \times W$, $R(s;\hat{\mu})$ extends to a $\mathscr{C}^{\lfloor L \rfloor}$ function which is $\lfloor L \rfloor$-flat with respect to s at $s = 0$.

Where do the compensators come from?



$$T_2(s) = \int_s^{\varepsilon} \left(y^n + u^{\ell} Q_\mu(u)\right)\Big|_{y=\delta(s/x)^\lambda} \frac{dx}{x}$$

$$= c_0 + c_1 s^{\lambda n} + \sum_{k=\ell}^{\ell+m} d_{k-\ell} s^{\lambda q k} \int_s^{\varepsilon} x^{(p-\lambda q)k} \frac{dx}{x}$$

Where do the compensators come from?

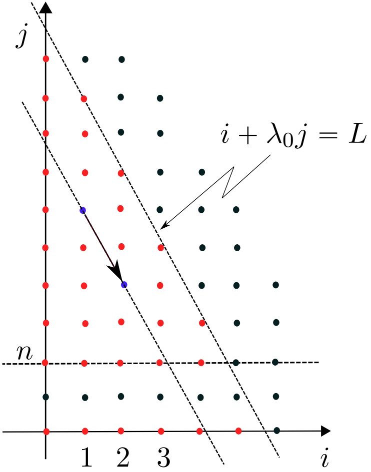
The monomials $s^{i_1+\lambda j_1}$ and $s^{i_2+\lambda j_2}$ are not well ordered if $i_1 + \lambda_0 j_1 = i_2 + \lambda_0 j_2$, which implies $\lambda_0 \in \mathbb{Q}$ and

$$(i_1, j_1) = (i_2, j_2) + r(p, -q)$$

for some $r \in \mathbb{Z}$ and $\lambda_0 = p/q$ with $\gcd(p, q) = 1$. Then

$$\begin{aligned} s^{i_2+\lambda j_2} &= s^{i_1+\lambda j_1} s^{-r(p-\lambda q)} \\ &= s^{i_1+\lambda j_1}(1 + \alpha\omega(s;\alpha))^r \end{aligned}$$

with $\alpha = p - \lambda q$ because recall that $\omega(s;\alpha) = \frac{s^{-\alpha}-1}{\alpha}$ for $\alpha \neq 0$.



The Dulac time

- If $f(s;\nu)$ and $g(s;\nu)$ are defined on $(0,\varepsilon)\times U$ for some open set U of $\mathbb{R}^N$ and $\varepsilon>0$, we write $f \prec_{\nu_0} g$ in case that

$$\lim_{(s,\nu)\to(0,\nu_0)} \frac{g(s;\nu)}{f(s;\nu)} = 0.$$

Observe that this is a strict partial order.

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  Observe that this is a strict partial order.

- If $\lambda_0 \in \mathbb{Q}$ with $\lambda_0 = p/q$ then

$$s^{i+\lambda j}\omega^k(s;p-\lambda q) \prec_{\lambda_0} s^{i'+\lambda j'}\omega^{k'}(s;p-\lambda q)$$

  in case that $i+\lambda_0 j < i'+\lambda_0 j'$ or $\{i=i',\ j=j' \text{ and } k>k'\}$.

The Dulac time

Theorem B provides the (ordered) list of monomials that can appear in the asymptotic development of $T(\cdot\,;\mu)$ near $s = 0$. Indeed, in case that $\lambda_0 \in \mathbb{Q}$, condition $(a3)$ implies that if $P_{ij}^{\lambda_0}$ and $P_{i'j'}^{\lambda_0}$ are not identically zero then

$$i + \lambda_0 j \neq i' + \lambda_0 j'.$$

In its turn this implies that $\prec_{\lambda_0}$ is a strict total order among the monomials $s^{i+\lambda j}\omega^k(s; p - \lambda q)$ that appear in the development.

The Dulac time

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In its turn this implies that $\prec_{\lambda_0}$ is a strict total order among the monomials $s^{i+\lambda j}\omega^k(s;p-\lambda q)$ that appear in the development. Setting

$$P_{ij}^{\lambda_0}(z;\hat{\mu})=\sum_{k=0}^{\ell_{ij}}\Delta_{ijk}(\hat{\mu})z^k \text{ with } \Delta_{ijk}\in\mathscr{C}^{\infty}\big(I_{ij}^{\lambda_0}\times W\big),$$

let $\{\Delta_d\}_{d\in\mathbb{N}}$ be the sequence of coefficients Δ_{ijk} (re)labelled according the position of $s^{i+\lambda j}\omega^k(s;p-\lambda q)$ in the list of monomials ordered with respect to $\prec_{\lambda_0}$.

The Dulac time

- Consider $\hat{\mu}_0 = (\lambda_0, \mu_0) \in (0, +\infty) \times W$ and let $\{\Delta_r\}_{r \geqslant 2}$ be the previous sequence of coefficients according to $\prec_{\lambda_0}$. We define

$$\ell_{\hat{\mu}_0} := \min\{r \geqslant 2 : \Delta_r(\hat{\mu}_0) \neq 0\} - 2.$$

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- Let $h(s; \hat{\mu})$ be a $\mathscr{C}^\infty$ function on $(0, \varepsilon) \times \hat{W}$ for some $\varepsilon > 0$. Given any $\hat{\mu}_0 \in U$ we define $\mathcal{Z}_0(h(\,\cdot\,; \hat{\mu}), \hat{\mu}_0)$ to be the smallest integer N having the property that there exist $\delta > 0$ and a neighbourhood V of $\hat{\mu}_0$ such that for every $\hat{\mu} \in V$ the function $h(s; \hat{\mu})$ has no more than N zeros on $(0, \delta)$ counted with multiplicities.

The Dulac time

Theorem C

Suppose that $\{Y_{\hat{\mu}}\}_{\hat{\mu}\in\hat{W}}$ is a family of vector fields with $n \in \mathbb{N}$ and verifying the FLP. Let $T(\,\cdot\,;\hat{\mu})$ be the Dulac time between the transverse sections Σ_σ and Σ_τ and fix some $\hat{\mu}_0 \in \hat{W}$. If $\ell_{\hat{\mu}_0}$ is finite then $\mathcal{Z}_0(T'(\,\cdot\,;\hat{\mu}),\hat{\mu}_0) \leqslant \ell_{\hat{\mu}_0}$.

Critical periodic orbits

- A singular point p of a smooth differential system is a center if it has a punctured neighbourhood that consists of periodic orbits surrounding p. The largest punctured neighbourhood with this property is called the period annulus of the center and it will be denoted by $\mathscr{P}$. Henceforth $\partial\mathscr{P}$ will denote the boundary of $\mathscr{P}$ after embedding it into $\mathbb{RP}^2$. Clearly the center p belongs to $\partial\mathscr{P}$, and in what follows we will call it the inner boundary of the period annulus. We also define the outer boundary of the period annulus to be $\Pi := \partial\mathscr{P}\setminus\{p\}$. Note that Π is a non-empty compact subset of $\mathbb{RP}^2$.

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- The period function of the center assigns to each periodic orbits in $\mathscr{P}$ its period. To study its qualitative properties usually the first step is to parametrize this set. This can be done by taking a smooth transverse section to the vector field on $\mathscr{P}$, for instance an orbit of the orthogonal vector field. If $\{\gamma_s\}_{s\in(0,1)}$ is such a parametrization, then $s \longmapsto P(s) := \{\text{period of } \gamma_s\}$ is a smooth map that provides the qualitative properties of the period function that we are interested in.

Critical periodic orbits

- The critical periods are isolated critical points of P, i.e. $\hat{s} \in (0,1)$ such that $P'(s) = \alpha(s-\hat{s})^k + \mathrm{o}\big((s-\hat{s})^k\big)$ with $\alpha \neq 0$ and $k \geqslant 1$. In this case we shall say that $\gamma_{\hat{s}}$ is a critical periodic orbit of multiplicity k of the center.

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- Consider a $\mathscr{C}^\infty$ family $\{X_\nu\}_{\nu \in U}$ of planar polynomial vector fields with a center and fix some $\nu_0 \in U$. Suppose that the outer boundary of the period annulus varies continuously at $\nu_0 \in U$, meaning that for any $\varepsilon > 0$ there exists $\delta > 0$ such that $d_H(\Pi_\nu, \Pi_{\nu_0}) \leqslant \varepsilon$ for all $\nu \in U$ with $\|\nu - \nu_0\| \leqslant \delta$. Then, setting

$$N(\delta,\varepsilon) = \sup\big\{\#\text{ critical periodic orbits } \gamma \text{ of } X_\nu \text{ in } \mathscr{P}_\nu \\ \text{with } d_H(\gamma, \Pi_{\nu_0}) \leqslant \varepsilon \text{ and } \|\nu-\nu_0\| \leqslant \delta\big\},$$

the criticality of (Π_{ν_0}, X_{ν_0}) with respect to the deformation X_ν is

$$\mathrm{Crit}\big((\Pi_{\nu_0}, X_{\nu_0}), X_\nu\big) := \inf_{\delta,\varepsilon} N(\delta,\varepsilon).$$

A criticality result

Using a result of [Mardešić and Saavedra, 2007] we obtain the following result which can be thought as the time counterpart of the finite cyclicity for a separatrix loop of a hyperbolic saddle ([Leontovich 1951], [Roussarie 1986] and [Yakovenko and Il'yashenko 1991]):

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Corollary C

Consider a $\mathscr{C}^\infty$ family of symmetric planar polynomial vector fields $\{X_\nu\}_{\nu\in U}$ with a center laying in the symmetry axis. Suppose moreover that the outer boundary Π_ν of its period annulus varies continuously and has only two singular points, which are hyperbolic saddle points verifying the FLP and not laying in the symmetry axis. Fix any $\nu_0 \in U$.

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