Rigorous Computer-Assisted Application of KAM Theory: A Modern Approach

J.Ll. Figueras [1] A. Haro [2] A. Luque [1]

[1] Uppsala Universitet (UU)

[2] Universitat de Barcelona (UB)

Segona Jornada de Sistemes Dinàmics a Catalunya

Barcelona, 4 d'octubre de 2017

Benvolguts i benvolgudes,

Primer de tot ens volem disculpar de no poder estar presents en aquestes jornades de sistemes dinàmics que tenen lloc a casa nostra i compartir el dia amb companys i companyes molt especials.

Volem donar les gràcies per l'honor que ens suposa rebre aquest premi. Volem accentuar aquestes gràcies a en Carles Simó, que sense la seva iniciativa d'establir el premi aquest no seria possible.

Moltíssimes gràcies i records des d'Uppsala!

Jordi-Lluís Figueras
Alejandro Luque

On computer assisted proofs and KAM theory

Computer assisted proofs is a very interesting area in which it is possible to find a meaningful collaboration between Mathematicians (proving theorems of the right kind), Computer Scientists (developing good software tools that relieve the tedium of programming the variants required) and Applied Scientists (that have challenging real life problems).

Rafael de la Llave

Computer aided proofs in dynamical systems back more than 30 years ago, and have lead to proofs of long-standing problems:

- Feigenbaum conjecture [Lanford 82][Lanford 84];
- Existence of Lorenz attractor [Tucker 99, 02].

Nowadays, there are research groups developing software:

- CAPD, Computer Assisted Proofs in Dynamics (P. Zgliczynski);
- CHOMP, Computational Homology Project (K. Mischaikov).

... and many researches making advances in the field (G. Arioli, H. Koch, J.P. Lessard, J. Mireles-James, J.B. Van Der Berg, Rafael de la Llave, Jordi-Lluís Figueras, etc.)

In this talk

The goal of this talk is to present a methodology to perform Computer Assisted Proofs in KAM theory, for Lagrangian tori for exact symplectic maps, based on the parameterization method [Figueras, Haro, Luque 16].

Main tools:

- KAM theorem in an *a posteriori* format [de la Llave 01], [González, Jorba, de la Llave, Villanueva 05],[Haro,Luque 16];
- FFT sharp bounds of analytic norms [Epstein 05];
- Assignation of Diophantine constants to interval vectors of frequencies;
- Sharper Rüssman estimates [Rüssmann 75, 76].

Applications:

- Validation of the golden torus for the standard map;
- Validation of meandering invariant tori.

For the standard map

$$\begin{array}{rcl} F_\varepsilon : \mathbb{T} \times \mathbb{R} & \longrightarrow & \mathbb{T} \times \mathbb{R} \\ (x, y) & \longmapsto & (x + y - \frac{\varepsilon}{2\pi} \sin(2\pi x), y - \frac{\varepsilon}{2\pi} \sin(2\pi x)). \end{array}$$

We take $\omega = (\sqrt{5} - 1)/2$.

For $\varepsilon = 0$, the torus parametrized by

$$K(\theta) = \begin{pmatrix} \theta \\ \omega \end{pmatrix}$$

is invariant, and its internal dynamics is a rotation by ω.

We continue the torus for $\varepsilon > 0$.

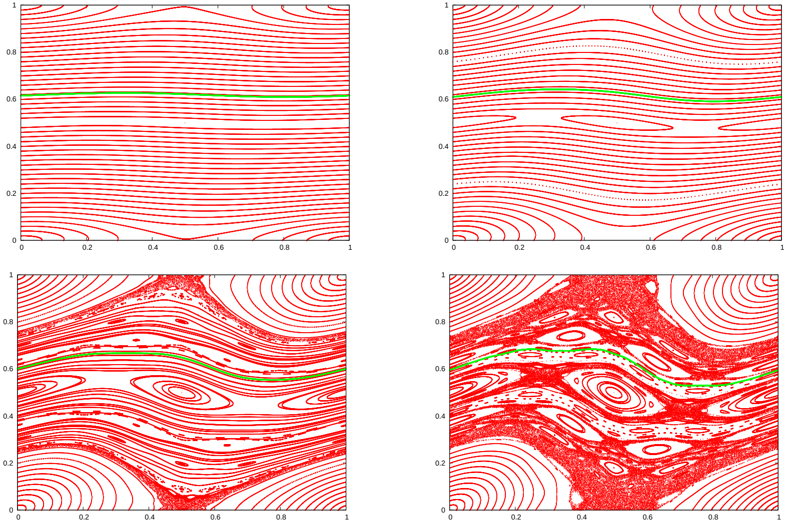


Figure : Phase portrait for $\varepsilon = 0.1, 0, 3, 0.7, 0.97$

Classical KAM methods are based on the use of canonical changes of coordinates. They typically consider perturbative problems and take advantage of the existence of action-angle coordinates.

The golden torus persist for ...

- $\varepsilon \leq 0.029$ [Herman 86] ;
- $\varepsilon \leq 0.68$ [Celletti, Chierchia 88];
- $\varepsilon \leq 0.91$ [de la Llave, Rana 90, 91].

The critical value is $\varepsilon_* \simeq 0.97163540$
(using e.g. Greene's method, blow up of Sobolev norm).

We know that for 0.9718 the curve does not exist [Jungreis 91].

A goal: to obtain persistence for $\varepsilon = 0.9716$.

We will use a posteriori KAM result, instead of a perturbative result.

See [Figueras, Haro 12] for validation of fiberwise hyperbolic invariant tori on the verge of breakdown in **non-autonomous** (quasi-periodic) dynamical systems.

An a posteriori KAM theorem

Exact symplectic maps and invariant tori

- Let $\mathcal{A} \subset \mathbb{T}^n \times \mathbb{R}^n$ be an annulus, with coordinates $z = (z_1, \ldots, z_{2n})$.
- $\mathcal{A}$ is endowed with an exact symplectic form $\boldsymbol{\omega} = \mathrm{d}\boldsymbol{\alpha}$. In coordinates:

$$a(z) = (a_1(z), \ldots, a_{2n}(z))^\top, \qquad \Omega(z) = \mathrm{D}a(z)^\top - \mathrm{D}a(z).$$

- An exact symplectomorphism $F : \mathcal{A} \to \mathcal{A}$ is a diffeomorphism such that $F^*\boldsymbol{\alpha} - \boldsymbol{\alpha} = \mathrm{d}S$ for a primitive function $S : \mathcal{A} \to \mathbb{R}$. In particular, $F^*\boldsymbol{\omega} = \boldsymbol{\omega}$.
- A torus $\mathcal{K}$, parameterized by $K : \mathbb{T}^n \to \mathcal{A}$, is F-invariant with frequency $\omega \in \mathbb{R}^n$ if

$$F \circ K = K \circ R_\omega,$$

  where $R_\omega(\theta) = \theta + \omega$.
- We assume F is homotopic to the identity, and K is homotopic to the zero section.
- We work in the real-analytic category.

KAM theorem (*a posteriori* format)

Given $\omega \in \mathbb{R}^n$ (γ, τ)-Diophantine: $(k, m) \in \mathbb{Z}^n \times \mathbb{Z}, k \neq 0 \Rightarrow |k \cdot \omega - m| \geq \dfrac{\gamma}{|k|_1^\tau}$.

Given $K : \mathbb{T}^n \to \mathcal{A}$, let $E : \mathbb{T}^n \to \mathbb{R}^n \times \mathbb{R}^n$ be the error function

$$E = F \circ K - K \circ R_\omega .$$

Assume that K satisfies certain non-degeneracy conditions.
Then, for every $0 < \rho_\infty < \rho_1 < \rho$ there exist constants $\mathfrak{C}_1$ and $\mathfrak{C}_2$ such that if

$$\frac{\mathfrak{C}_1 \|E\|_\rho}{\gamma^4 \rho^{4\tau}} < 1,$$

then there exists a F-invariant torus $\mathcal{K}_\infty = K_\infty(\mathbb{T}^n)$, with frequency ω, analytic in $\mathbb{T}^n_{\rho_\infty}$ and continuous in $\bar{\mathbb{T}}^n_{\rho_\infty}$, and close to the initial approximate solution:

$$\|K_\infty - K\|_{\rho_\infty} \leq \frac{\mathfrak{C}_2}{\gamma^2 \rho^{2\tau}} \|E\|_\rho .$$

Hypothesis 1: The 2-form ω, the 1-form α and the map F are real-analytic and can be holomorphically extended to some complex strip $\mathcal{B}$, and such that

$$\|\Omega\|_{\mathcal{B}} \leq c_{\Omega,0}, \qquad \|\mathrm{D}\Omega\|_{\mathcal{B}} \leq c_{\Omega,1},$$
$$\|\mathrm{D}a\|_{\mathcal{B}} \leq c_{a,1}, \qquad \|\mathrm{D}^2 a\|_{\mathcal{B}} \leq c_{a,2},$$
$$\|\mathrm{D}F\|_{\mathcal{B}} \leq c_{F,1}, \qquad \|\mathrm{D}^2 F\|_{\mathcal{B}} \leq c_{F,2}.$$

There is a bundle $\mathcal{N}^0$, parameterized by $N^0 : \mathbb{T}^n \to \mathbb{R}^{2n\times n}$, where N^0 can be holomorphically extended to $\mathbb{T}^n_\rho$, for some $\rho > 0$, and continuously to $\bar{\mathbb{T}}^n_\rho$, with

$$\|N^0\|_\rho \leq c_{N^0}, \qquad \|(N^0)^\top\|_\rho \leq c^*_{N^0}.$$

Hypothesis 2: The torus K can be holomorphically extended to $\mathbb{T}^n_\rho$, and continuously to $\bar{\mathbb{T}}^n_\rho$, with $K(\bar{\mathbb{T}}^n_\rho) \subset \mathcal{B}$ and

$$\|\mathrm{D}K\|_\rho < \sigma_L, \qquad \|\mathrm{D}K^\top\|_\rho < \sigma^*_L, \qquad \mathrm{dist}(K(\bar{\mathbb{T}}^n_\rho), \partial\mathcal{B}) > 0,$$

Note: All the norms are sup-norms.

Hypothesis 3 [transversality condition]: The $n \times n$-matrix valued map

$$G(\theta) = -\mathrm{D}K(\theta)^\top \Omega(K(\theta)) N^0(\theta)$$

is non-singular, and $\quad \|G^{-1}\|_\rho < \sigma_G, \qquad \|G^{-\top}\|_\rho < \sigma_G^*.$

Hence, the $2n \times 2n$-matrix valued map

$$P(\theta) = \begin{pmatrix} \mathrm{D}K(\theta) & N(\theta) \end{pmatrix},$$

with

$$N(\theta) = L(\theta)A(\theta) + N^0(\theta)B(\theta),$$

$$B(\theta) = G(\theta)^{-1}, \; A(\theta) = -\frac{1}{2} B(\theta)^\top N^0(\theta)^\top \Omega(K(\theta)) N^0(\theta) B(\theta),$$

is non-singular.

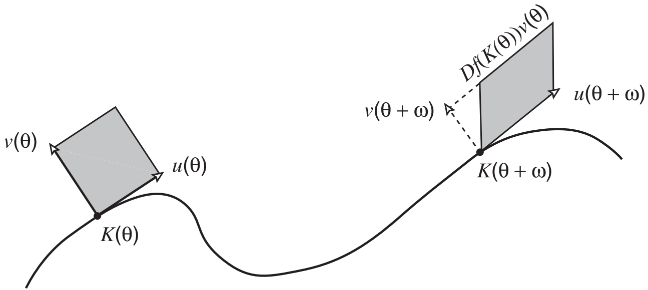
Hypothesis 4 [twist condition]: The average of the $n \times n$-matrix valued map

$$T(\theta) = N(\theta+\omega)^\top \, \Omega(K(\theta+\omega)) \, \mathrm{D}F(K(\theta)) \, N(\theta).$$

is invertible, and $\quad |\, \langle T \rangle^{-1} \,| < \sigma_T.$

The torsion T measures the twist of the complementary normal bundle N.

The torus is twist if the torsion is non-degenerate.



The proof of the KAM theorems consist of proving the convergence of a Newton-like method, in which the analyticity strip of the objects is reduced at each step.

From

- the initial analyticity strip $\rho = \rho_0$,
- the second analyticity strip $\rho_1 = \rho - \delta$, with $\delta_0 = \delta$,
- the limiting analyticity strip ρ_∞,

the intermediate analyticity strips are given by the recurrence

- $\rho_s = \rho_{s-1} - 3\delta_{s-1}$,
- $\delta_s = \frac{\delta_{s-1}}{a_1}$.

with

$$a_1 = \frac{\rho - \rho_\infty}{\rho_1 - \rho_\infty} .$$

The proof is constructive and provides explicit formulae for the constants $\mathfrak{C}_1$ and $\mathfrak{C}_2$.

The proof relies in delicate interactions between Geometry (symplectic properties) and Analysis (small divisors problems).

Towards the CAP-KAM

To-do list

- Task1: To obtain FAST, rigorous and sharp estimates of analitic norms for functions depending on multiple angles, using FFT.
- Task 2: To assign Diophantine constants to a frequency vector. This vector may be given with finite precision.
- Task 3: To obtain sharp Rüssmann's estimates that improve the applicability of the KAM theorem.

E.L. Epstein. How well does the finite Fourier transform approximate the Fourier transform? CPAM 2005.

Theorem

Let $f : \mathbb{T}^n_{\hat\rho} \to \mathbb{C}$ be a holomorphic and bounded function in the complex strip $\mathbb{T}^n_{\hat\rho}$ of size $\hat\rho > 0$. Let $\tilde f$ be the discrete Fourier approximation of f in the regular grid of size $N = (N_1, \ldots, N_n)$. Then, for $0 \leq \rho < \hat\rho$:

$$\|\tilde f - f\|_\rho \leq C_N(\rho, \hat\rho)\|f\|_{\hat\rho},$$

where $C_N(\rho, \hat\rho) \to 0$ when all $N_\ell \to \infty$.

[Epstein 05] considers the case $n = 1$ and $\rho = 0$.

Corollaries: rigorous error bounds in FFT products and inverses.

Consequences: the computational cost of the computations is $O(N_D \log N_D)$, where $N_D = N_1 \ldots N_n$.

Proposition

Let $\varpi = \prod_{i=1}^n [a_i, b_i]$ *be an interval vector. Let* $M \geq n$ *be such that for any* $\omega \in \varpi$, $k \in \mathbb{Z}^n$ *with* $0 < |k|_1 \leq M$, *and* $m \in \mathbb{Z}$, *we have* $k \cdot \omega - m \neq 0$. *For any* $\tau > n$, *we define*

$$\gamma_M(\varpi, \tau) = \min\{|k \cdot \omega - m||k|_1^\tau \,:\, \omega \in \varpi,\ 0 < |k|_1 \leq M,\ m \in \mathbb{Z}\}.$$

Then, for any positive $\gamma \leq \gamma_M(\varpi, \tau)$, *we have that the relative measure of the set of* (γ, τ)*-Diophantine vectors in* ϖ *is*

$$p(\varpi, \gamma, \tau) := 1 - \frac{\mathrm{meas}\,(\mathrm{Res}(\varpi, \gamma, \tau))}{\mathrm{meas}(\varpi)} > 1 - \frac{C(\varpi, n)\gamma}{(\tau - n)M^{\tau - n}} \geq 1 - \frac{C(\varpi, n)\gamma_M(\varpi, \tau)}{(\tau - n)M^{\tau - n}},$$

where $C(\varpi, n) = \dfrac{2^{2n}}{(n-1)!}\dfrac{(\mathrm{diam}(\varpi))^n}{\mathrm{meas}(\varpi)}$.

Moreover, the equation for τ

$$(\tau - n)M^{\tau - n} - C(\varpi, n)\gamma_M(\varpi, \tau) = 0$$

has a unique solution $\tau_M(\varpi)$. *As a consequence, for any pair* (γ, τ) *with* $\tau \geq \tau_M(\varpi)$ *and* $\gamma \leq \gamma_M(\varpi, \tau_M(\varpi))$, *we have* $p(\varpi, \gamma, \tau) > 0$.

$$\omega_{a,b} = \frac{\sqrt{b^2 + 4b/a} - b}{2}, \qquad \left[\omega_{a,b} - 2^{-50}, \omega_{a,b} + 2^{-50}\right] \subset \varpi.$$

a	b	$\gamma \leq$	$\tau \geq$
1	1	0.381966011250104	1.26
1	2	0.267949192431121	1.23
1	3	0.208712152522079	1.21
1	4	0.171572875253808	1.19
1	5	0.145898033750314	1.18
1	6	0.127016653792582	1.17
2	1	0.366025403784437	1.26
2	2	0.413767832000904	1.27
2	3	0.300011472016747	1.24
2	4	0.235323972166368	1.22
2	5	0.192798030208926	1.20
2	6	0.163806299636515	1.19

Lemma [sharper Rüssmann estimates]

Assume DC, analyticity and the equation $u(\theta) - u(\theta+\omega) = v(\theta) - \langle v \rangle$. Then, the only zero-average solution satisfies

$$\|u\|_{\rho-\delta} \le \frac{c_R(\delta)}{\gamma\delta^\tau}\|v\|_\rho,$$

with

$$c_R(\delta) = \sqrt{\sum_{0<|k|_1\le L} \frac{\gamma^2\delta^{2\tau}2^n \mathrm{e}^{-4\pi|k|_1\delta}}{4|\sin(\pi k\cdot\omega)|^2} + \frac{2^{n-3}\zeta(2,2^\tau)}{(2\pi)^{2\tau}} \int_{4\pi\delta(L+1)}^\infty u^{2\tau} e^{-u}\,\mathrm{d}u},$$

where $\zeta(a,b) = \sum_{j\ge 0}(b+j)^{-a}$ is the Hurwitz zeta function.

Remark:

$$c_R(\delta) < c_R := \frac{\sqrt{2^{n-3}\zeta(2,2^\tau)\Gamma(2\tau+1)}}{(2\pi)^\tau}.$$

(c_R follows from [Rüssmann 75,76]).

$\omega_{a,b}$		$c_R(\delta) \leq$					
a	b	$L = 0$	$\delta = 0.1$	$\delta = 0.01$	$\delta = 0.001$	$\delta = 0.0001$	$\delta = 0.00001$
1	1	6.53e-02	1.70e-02	1.01e-02	5.57e-03	3.06e-03	1.68e-03
1	2	6.76e-02	1.40e-02	8.84e-03	5.26e-03	3.08e-03	1.79e-03
1	3	6.92e-02	1.29e-02	8.71e-03	5.18e-03	3.29e-03	1.98e-03
1	4	7.09e-02	1.25e-02	8.36e-03	5.84e-03	3.60e-03	2.21e-03
1	5	7.18e-02	1.23e-02	7.88e-03	5.47e-03	3.93e-03	2.62e-03
1	6	7.27e-02	1.24e-02	8.01e-03	5.19e-03	3.59e-03	2.59e-03
2	1	6.53e-02	1.74e-02	1.04e-02	5.78e-03	3.21e-03	1.75e-03
2	2	6.46e-02	1.87e-02	1.09e-02	5.85e-03	3.14e-03	1.68e-03
2	3	6.68e-02	1.63e-02	9.88e-03	5.56e-03	3.12e-03	1.77e-03
2	4	6.84e-02	1.53e-02	9.71e-03	5.85e-03	3.53e-03	2.12e-03
2	5	7.01e-02	1.50e-02	9.60e-03	5.78e-03	3.43e-03	2.04e-03
2	6	7.09e-02	1.48e-02	9.04e-03	5.05e-03	2.95e-03	1.98e-03

The CAP-KAM algorithm

Given a parameterization K (as a truncated Fourier series), approximately invariant, we define, for $\sigma > 1$,

$$\sigma_L = \|\mathrm{D}K\|_\rho \sigma,\ \sigma_L^* = \|\mathrm{D}K^\top\|_\rho \sigma,\ \sigma_G = \|B\|_\rho \sigma,\ \sigma_G^* = \|B^\top\|_\rho \sigma, \sigma_T = |\left\langle T \right\rangle^{-1}| \sigma.$$

We take $\rho_\infty = 0$.
We have to select values of ρ, δ, σ, $\tilde{\rho}$ that would satisfy

$$\frac{\mathfrak{C}_1 \|E\|_\rho}{\gamma^4 \rho^{4\tau}} < 1.$$

In order to control the global objects we introduce the domain

$$\mathcal{B} = \{(x,y) \in \mathbb{C}^n/\mathbb{Z}^n \times \mathbb{C}^n \; : \; |\mathrm{im}\, x_i| \leq d_{\mathcal{B}} + \rho + \|K_p^{x_i}\|_{F,\rho} \,,\; |y_i| \leq d_{\mathcal{B}} + \|K_p^{y_i}\|_{F,\rho}\},$$

and in order to control FFT approximations we introduce the domain

$$\hat{\mathcal{B}} = \{(x,y) \in \mathbb{C}^n/\mathbb{Z}^n \times \mathbb{C}^n \; : \; |\mathrm{im}\, x_i| \leq \hat\rho + \|K_p^{x_i}\|_{F,\hat\rho} \,,\; |y_i| \leq \|K_p^{y_i}\|_{F,\hat\rho}\}.$$

We compute the upper estimates

$$\|\mathrm{D}a\|_{\mathcal{B}} \leq c_{a,1}, \quad \|\mathrm{D}^2 a\|_{\mathcal{B}} \leq c_{a,2}, \quad \|\Omega\|_{\mathcal{B}} \leq c_{\Omega,0}, \quad \|\mathrm{D}\Omega\|_{\mathcal{B}} \leq c_{\Omega,1},$$

$$\|\mathrm{D}F\|_{\mathcal{B}} \leq c_{F,1}, \qquad \|\mathrm{D}^2 F\|_{\mathcal{B}} \leq c_{F,2},$$

which appear in the KAM theorem, and the upper estimates

$$\|\Omega\|_{\hat{\mathcal{B}}} \leq \hat c_{\Omega}, \qquad \|F_p\|_{\hat{\mathcal{B}}} \leq \hat c_{F_p}.$$

In the standard case, $c_{\Omega,0} = 1$, $c_{\Omega,1} = 0$, $c_{a,1} = 1$ and $c_{a,2} = 0$.

The initial parameterization satisfies the invariance equation up to an error

$$E(\theta) = \begin{pmatrix} K_p^x(\theta) + F_p^x(K(\theta)) - K_p^x(\theta+\omega) - \omega \\ F_p^y(K(\theta)) - K_p^y(\theta+\omega) \end{pmatrix}.$$

For instance, for the standard map

$$E(\theta) = \begin{pmatrix} K_p^x(\theta) + \frac{\varepsilon}{2\pi}\sin(2\pi K^x(\theta)) - K_p^x(\theta+\omega) - \omega \\ \frac{\varepsilon}{2\pi}\sin(2\pi K^x(\theta)) - K_p^y(\theta+\omega) \end{pmatrix}.$$

Notice that the difficult part is $\sin(2\pi K^x(\theta))$ that contains infinitely many Fourier modes.

Using our approximation theorem (task 1), since K_p equals its DFT $\tilde{K}_p$, the error

$$E(\theta) = \begin{pmatrix} K_p^x(\theta) + F_p^x(K(\theta)) - \omega - K_p^x(\theta+\omega) \\ F_p^y(K(\theta)) \qquad\quad - K_p^y(\theta+\omega) \end{pmatrix}$$

can be bounded from its DFT $\tilde{E}$ by

$$\|E\|_\rho \leq \|\tilde{E}\|_\rho + C_N(\rho, \hat{\rho}) \|F_p\|_{\hat{\mathcal{B}}},$$

where we take $0 < \rho < \hat{\rho}$.

Similar ideas lead to obtain rigorous upper bounds to validate the non-degeneracy conditions, from discrete Fourier transforms.

We can validate the rest of the hypotheses of the KAM theorem, and check the condition on the error bounds.

The consequence is that the CAP has a cost $O(N_D \log(N_D))$.

Example 1: The standard map and the golden curve

The standard map and the golden curve

We consider the standard map

$$\begin{array}{rcl} F_\varepsilon : \mathbb{T}\times\mathbb{R} & \longrightarrow & \mathbb{T}\times\mathbb{R} \\ (x,y) & \longmapsto & \left(x+y-\dfrac{\varepsilon}{2\pi}\sin(2\pi x), y-\dfrac{\varepsilon}{2\pi}\sin(2\pi x)\right). \end{array}$$

We continue with respect to ε the torus K with rotation vector $\omega = \frac{1}{2}\sqrt{5}-1$.

We consider the transversal bundle $N_0(\theta) = (0,1)$.

We use $\gamma = \frac{3-\sqrt{5}}{2}$, $\tau = 1$ and the *ad hoc* Rüssmann estimates.

ε	N_{D}	ρ	δ	$\sigma - 1$	$d_{\mathcal{B}}$	$\hat{\rho}$	$\frac{\mathfrak{C}_1 b_E}{\gamma^4 \rho^{4\tau}}$	$\frac{\mathfrak{C}_2 b_E}{\gamma^2 \rho^{2\tau}}$
0.06	128	1.606160e-02	3.212319e-03	1.670325e-01	5.064098e-06	2.569855e-01	1.35e-28	9.47e-34
0.16	256	1.369960e-02	2.739919e-03	9.673976e-02	2.937365e-06	1.369960e-01	9.24e-28	3.77e-33
0.26	256	1.369960e-02	2.739919e-03	6.974093e-02	2.044422e-06	1.301462e-01	1.74e-26	4.94e-32
0.36	512	1.369960e-02	2.739919e-03	5.229422e-02	1.400906e-06	7.534778e-02	4.24e-25	8.26e-31
0.46	512	1.369960e-02	2.739919e-03	3.941981e-02	9.278480e-07	7.534778e-02	1.76e-23	2.27e-29
0.56	512	4.520867e-03	8.908112e-04	1.268703e-02	9.401294e-08	6.329214e-02	9.39e-24	1.24e-30
0.66	1024	3.300233e-03	5.973272e-04	1.047736e-02	4.061043e-08	3.300233e-02	1.88e-23	1.11e-30
0.76	1024	2.310163e-03	4.017675e-04	5.924431e-03	1.166394e-08	3.003212e-02	2.32e-18	3.98e-26
0.86	2048	1.178183e-03	1.996921e-04	1.921375e-03	1.234843e-09	1.531638e-02	1.74e-17	3.19e-26
0.96	32768	1.178183e-04	1.971855e-05	3.648874e-05	5.996316e-13	1.060365e-03	2.34e-12	2.09e-24

For $\varepsilon = 0.96$, the computation takes 117 seconds, using interval arithmetic with 267 bits (80 digits).

With the *ad hoc* Rüssmann estimates we obtain

$$\frac{\mathfrak{C}_1 b_E}{\gamma^4 \rho^{4\tau}} \leq 2.34 \cdot 10^{-12}, \qquad \frac{\mathfrak{C}_2 b_E}{\gamma^2 \rho^{2\tau}} \leq 2.09 \cdot 10^{-24};$$

With the *classical* Rüssmann estimates we obtain

$$\frac{\mathfrak{C}_1 b_E}{\gamma^4 \rho^{4\tau}} \leq 5.42 \cdot 10^{-6}, \qquad \frac{\mathfrak{C}_2 b_E}{\gamma^2 \rho^{2\tau}} \leq 1.71 \cdot 10^{-21}.$$

Theorem

For $\varepsilon = 0.9716$*, the standard map has a golden invariant curve.*

Proof.

We compute K using the parameterization method with $N_{\mathrm{D}} = 8388608$ Fourier coefficients, and this parameterization satisfies $\|E\|_0 \leq 2.74 \cdot 10^{-41}$. Setting the parameters, we take

$$\rho = 3.748290 \cdot 10^{-7}, \qquad \delta = 6.273289 \cdot 10^{-8}, \qquad \sigma - 1 = 1.610158 \cdot 10^{-9},$$

$$d_{\mathcal{B}} = 3.159428 \cdot 10^{-21}, \qquad \hat{\rho} = 4.872777 \cdot 10^{-6},$$

to apply the validation algorithm with precision of 367 bits and obtain (after 11404 seconds in a single processor Intel(R) Core(R) CPU at 3.50 GHz):

$$\frac{\mathfrak{C}_1 b_E}{\gamma^4 \rho^{4\tau}} \leq 0.0823.$$

The proof follows from the KAM theorem. Moreover, the golden curve satisfies

$$\|K_\infty - K\|_{\rho_\infty} < \frac{\mathfrak{C}_2 b_E}{\gamma^2 \rho^{2\tau}} \leq 3.89 \cdot 10^{-22}.$$

□

The golden curve for 0.9716

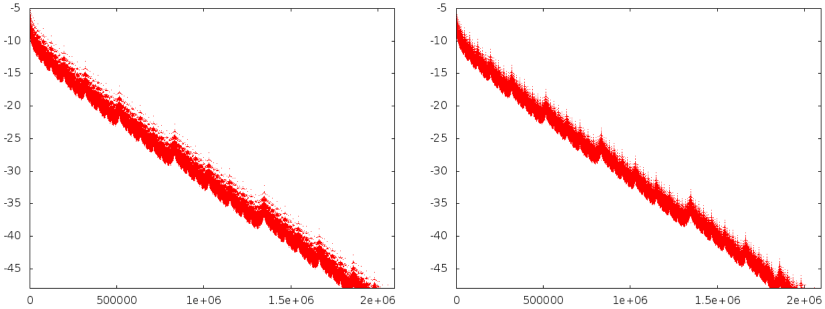


Figure : Fourier modes of the validated parameterization of the golden invariant curve for $\varepsilon = 0.9716$: (left) $k \mapsto \log_{10}(|K^x_{p,k}|)$; (right) $k \mapsto \log_{10}(|K^y_{p,k}|)$.

Example 2: Validation of a meandering invariant curve

We consider the non-twist standard map

$$\begin{array}{rcl} F : \mathbb{T}\times\mathbb{R} & \longrightarrow & \mathbb{T}\times\mathbb{R} \\ (x,y) & \longmapsto & (\bar{x},\bar{y}) = \left(x + (\bar{y}+\lambda_1)(\bar{y}+\lambda_2),\, y - \frac{\varepsilon}{2\pi}\sin(2\pi x)\right) \end{array}$$

with parameters $\lambda_1 = 0.1$, $\lambda_2 = -0.2$ and $\varepsilon = 0.45$.

It (apparently) has a meandering invariant curve with $\omega = \frac{\sqrt{5}-1}{32}$.
The torsion is $\langle T\rangle \simeq 0.0309003$.

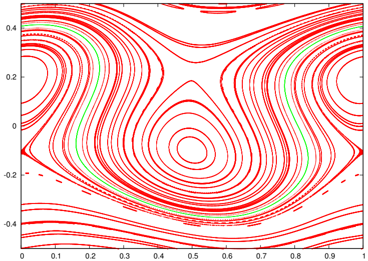


Figure : Meandering invariant curves for $\lambda_1 = 0.1$, $\lambda_2 = -0.2$ and $\varepsilon = 0.45$.

Theorem

For $\lambda_1 = 0.1$, $\lambda_2 = -0.2$ and $\varepsilon = 0.45$, the non-twist standard map has a meandering invariant torus with rotation number $\omega = \frac{\sqrt{5}-1}{32}$.

Proof.

We compute K with $N_{\mathrm{D}} = 2048$ and $\|E\|_0 \leq 10^{-42}$.
Rigorous computations are performed using interval arithmetics with 267 bits.
We enclose ω in a tight interval of length 10^{-80} (to compute γ, τ).
We select parameters

$$\rho = 1.223945 \cdot 10^{-3}, \qquad \delta = 2.048444 \cdot 10^{-4}, \qquad \sigma - 1 = 1.601973 \cdot 10^{-11},$$

$$d_{\mathcal{B}} = 8.333835 \cdot 10^{-22}, \qquad \hat{\rho} = 1.835918 \cdot 10^{-2}.$$

We use our *ad hoc* Rüssmann estimates.
The obtained result is (after 65 seconds in a single processor Intel(R) Xeon(R) CPU at 2.40 GHz),

$$\frac{\mathfrak{C}_1 b_E}{\gamma^4 \rho^{4\tau}} \leq 0.0343, \qquad \frac{\mathfrak{C}_2 b_E}{\gamma^2 \rho^{2\tau}} \leq 3.78 \cdot 10^{-23},$$

and the proof follows from the KAM theorem. □

Conclusions and future work

Conclusions

- We have presented a general KAM result with very sharp and explicit estimates for all the objects involved.
- The proof results in an efficient numerical method to compute invariant tori that we have implemented in a very general and flexible way.

A. Haro, M. Canadell, J.Ll. Figueras, A. Luque, J.M. Mondelo, *The parameterization method for invariant manifolds: from rigorous results to efective computations*, Applied Mathematical Sciences 195, Springer (2016).

- Numerical computations can be rigorously validated using the KAM theorem.
- The methology works in the far-from-integrable regime.
- We have also results for 2D tori (Froeschlé map).

J.Ll. Figueras, A. Haro. A. Luque, *Rigorous Computer-Assisted Application of KAM Theory: A Modern Approach*, Foundations of Computational Mathematics (2016).

Future work

- We plan to apply the CAP methodology to higher dimensions and consider more complex problems, e.g. in Celestial Mechanics.

  There are CAP of KAM tori in Celestial Mechanics by using classical methods in a very delicate way [Celletti, Chierchia 97],[Locatelli 98],[Locatelli, Giorgilli 00, 05].

- We pretend to obtain estimates on the measure of invariant tori in phase space.

- We are currently working in computing rigorous and realistic lower bounds of measure in parameter space for reducibility of families of analytic cercle maps.

- We plan also to consider the problem of computing rigorous and realistic lower bounds of measure of spectrum of quasi-periodic Schrödinger operators.

Prediction is very difficult, especially about the future.

Niels Bohr

Moltes gràcies !