Parabolic manifolds and the parameterization method

Inma Baldomá[1], Ernest Fontich[2], Pau Martín[1]

[1] Dept. Matemàtiques, Universitat Politècnica de Catalunya
[2] Dept. Matemàtiques i Informàtica, Universitat de Barcelona

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Plan

- Introduction, 2-dimensional maps
- The parameterization method
- 1-dimensional stable sets
- n-dimensional stable sets

  I Baldomá; EF ; P Martín:
  Invariant manifolds of parabolic fixed points (I). Existence and dependence on parameters. ArXiv:1603.02533

  Invariant manifolds of parabolic fixed points (II). Approximations by sums of homogeneous functions. ArXiv:1603.02535

- Gevrey estimates for 1 dimensional stable sets.

  I Baldomá; EF ; P Martín: Work in progress

We consider maps $F : U \subset \mathbb{R}^n \to \mathbb{R}^n$ with

$$F(0) = 0, \qquad DF(0) = \mathrm{Id}\,.$$

A neighbourhood of 0 is the center manifold.

Case $n = 1$

$$F(x) = x + a_k x^k + \dots, \qquad a_k \neq 0$$

Takens 73: If $F \in C^\infty$, it is C^∞ conjugate to $F(x) = x \pm x^k + \delta x^{2k-1}$

Case $n = 2$

Unstabilty results go back to Levi-Civita, 1901

Characterizations of stability, C. Simó, 1980's

Case $n = 1$ complex An example: 1D complex maps

$$F : z \mapsto z + az^k + \dots$$

Leau Fatou flower theorem

Taking $k = 2$ and $a = -1$

$$F\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} x - x^2 + y^2 \\ y - 2xy \end{pmatrix}$$

We observe that the stable sets may be

- open sets,
- curves with one extreme at the fixed point.

Let $U_r^+ = [0, r) \times (-r, r)$. We define

$$W_r^{s+} = \{z \in \mathbb{R}^2 \mid F^n(z) \in U_r^+, \forall n \geq 0, \quad \lim_{n\to\infty} F^n(z) = 0\},$$

$$W_r^{u+} = \{z \in \mathbb{R}^2 \mid F^{-n}(z) \in U_r^+, \forall n \geq 0, \quad \lim_{n\to\infty} F^{-n}(z) = 0\}.$$

$$F\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} x + p(x,y) + f(x,y) \\ y + q(x,y) + g(x,y) \end{pmatrix}$$

Slotnick (1958) considered Poincaré maps of analytic periodic Hamiltonian systems $H(x, y, t)$

McGehee (1973) considered non-conservative analytic maps, with p, q homogeneous polynomials of degree N and

- $p(x, 0) < 0$
- $q(x, 0) = 0$
- $q_y(x, 0) > 0$

He proved that $\exists \beta, r$ s.t.

$$W_r^{s+} \cap \{(x, y);\, 0 < x < r, |y| < \beta x\} = \operatorname{graph} \varphi$$

with φ differentiable and $\varphi_{|(0,r)}$ analytic.

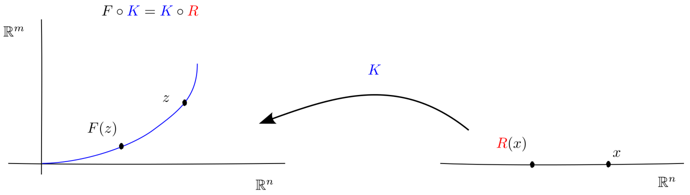
The parameterization method for maps

The parameterization method for I.M. of fixed points of maps $F : U \subset \mathbb{R}^{n+m} \to \mathbb{R}^{n+m}$, consists of looking for W as images of parameterizations. More concretely, if $F(0) = 0$ and we suspect that W is going to be tangent to $\mathbb{R}^n \times \{0\}$ we look for $K : V \subset \mathbb{R}^n \to \mathbb{R}^{n+m}$ such that $K(0) = 0$, $DK(0) = (\mathrm{Id}\,, 0)$, and

- K conjugates $F_{|W}$ to a possibly simpler map R in $\mathbb{R}^n$:

$$F \circ K = K \circ R.$$

Note that, the conjugacy implies $DF(0)(\mathrm{Id}\,, 0) = DR(0)$



The parameterization method for flows

- **The autonomous case**, $\dot{x} = X(x)$. We look for K (the parameterization) and Y (a new vector field) such that

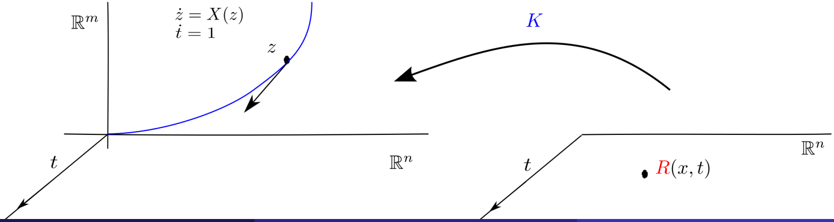
$$X(K(x)) = DK(x)Y(x).$$

- **The non-autonomous case**, $\dot{x} = X(x,t)$. We extend the vector field

$$\dot{x} = X(x,t), \qquad \dot{t} = 1$$

and apply the method of the autonomous case to obtain the invariance condition

$$X(K(x,t),t) = DK(x,t)Y(x,t) + \partial_t K(x,t).$$



The origin of the method can be traced back to the works of Poincaré and Lyapunov.

C. Simó. On the analytical and numerical approximations of invarinat manifolds, 1990

X Cabré; E F; R de la Llave. The parameterization method for invariant manifolds. I. Manifolds associated to non-resonant subspaces. Indiana Univ. Math. J. 52 (2003), no. 2, 283-328.

X Cabré; E F; R de la Llave. The parameterization method for invariant manifolds. III. Overview and applications. J. Differential Equations 218 (2005), no. 2, 444-515.

$+$ many papers, most of them from people in BCN:

Haro, González, Villanueva, Sire, Figueres, Mondelo, Luque, Canadell, Llave, Baldomá, Martín, ...

À Haro; M Canadell; JL Figueras; A Luque; JM Mondelo. The Parameterization Method for Invariant Manifolds. From Rigorous Results to Effective Computations, Springer, 2016.

One-dimensional submanifolds for differentiable maps

Let $F : U \subset \mathbb{R}^{1+m} \to \mathbb{R}^{1+m}$, with $F(0) = 0$, $DF(0) = \mathrm{Id}$.
We use the parameterization method to find an embedding $K : I \subset \mathbb{R} \to \mathbb{R}^{1+m}$ and a map $R : I \subset \mathbb{R} \to \mathbb{R}$ such that

$$F \circ K = K \circ R.$$

The fact that the curve passes through the origin and is tangent to the first axis is ensured by the supplementary conditions

$$K(0) = 0, \qquad DK(0) = (1, 0)^\top.$$

Theorem

[Baldomá, F., de la Llave, Martín, 05] Let $F : U \subset \mathbb{R}^{1+m} \to \mathbb{R}^{1+m}$ *be a* C^r *map,* $r \geq 2$ *or* $r = \infty$*, such that* $F(0,0) = 0$*,* $DF(0,0) = \mathrm{Id}$

$$\begin{cases} F_1(x,y) = & x \quad -ax^N + x^{N-1}\langle v, y\rangle + \hat{f}_N(x,y) \quad +f_{\geq N+1}(x,y), \\ F_2(x,y) = & y \quad \qquad\qquad +x^{M-1}By + \hat{g}_M(x,y) \quad +g_{\geq M+1}(x,y), \end{cases}$$

for some $2 \leq N, M \leq r$*. In the case that* $M \leq N$ *assume furthermore*

$$\mathrm{Spec}\, B \subset \{\lambda \in \mathbb{C} \mid \mathrm{Re}\,\lambda > 0\}.$$

Let $L = \min(N, M)$ *and* $\eta = 1 + N - L$*. We assume that* $r > 2N - 1$*.*

Theorem

Then there exist a C^p map $K : [0, t_0) \subset \mathbb{R} \to \mathbb{R}^{1+m}$, with $p = [(r - N + 1)/\eta] - 1$, of class C^r in $(0, t_0)$ and a polynomial $R : \mathbb{R} \to \mathbb{R}$ such that

$$F \circ K = K \circ R.$$

Moreover

$$K(t) = (t, 0) + O(t^2), \qquad R(t) = t - at^N + bt^{2N-1}$$

I Baldomá; E F; R de la Llave; P Martín. The parameterization method for one-dimensional invariant manifolds of higher dimensional parabolic fixed points. Discrete Contin. Dyn. Syst. 17 (2007), no. 4, 835-865.

n dimensional parabolic manifolds. Setting

Function sets:

$$\mathcal{H}^{\ell} = \{h : \mathcal{U} \to \mathbb{R}^k \ : \ \mathcal{C}^0, \text{ homogeneous of degree } \ell\}$$
$$\mathcal{H}^{\geq \ell} = \{h : \mathcal{U} \to \mathbb{R}^k \ : \ \mathcal{C}^0, \ \|h(x)\| = \mathcal{O}(\|x\|^{\ell})\}$$
$$\mathcal{H}^{> \ell} = \{h : \mathcal{U} \to \mathbb{R}^k \ : \ \mathcal{C}^0, \ \|h(x)\| = o(\|x\|^{\ell})\}.$$

We consider maps of the form

$$F\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} x + p(x,y) + f(x,y) \\ y + q(x,y) + g(x,y) \end{pmatrix}, \qquad \begin{array}{l} x \in \mathbb{R}^n \\ y \in \mathbb{R}^m \end{array}$$

with

$$p \in \mathcal{H}^N, \quad q \in \mathcal{H}^M, \quad f \in \mathcal{H}^{\geq N+1}, \quad g \in \mathcal{H}^{\geq M+1}$$

and

$$N, M \geq 2$$

The stable set

We will work with and open star shaped w.r.t. 0 set $V \subset \mathbb{R}^n$ such that $0 \in \partial V \cup V$.

In the analytic case, we will have to work in a complex neighborhood of V:

$$\Omega(\gamma) = \{z \in \mathbb{C}^n : \mathrm{Re} z \in V, \ \|\mathrm{Im} z\| \leq \gamma \|\mathrm{Re} z\|\}.$$

For technical reasons, we also have to consider

$$V_\rho = V \cap B_\rho(0), \qquad \Omega(\gamma, \rho) = \Omega(\gamma) \cap B_\rho(0).$$

Then we will deal with stable sets depending on V:

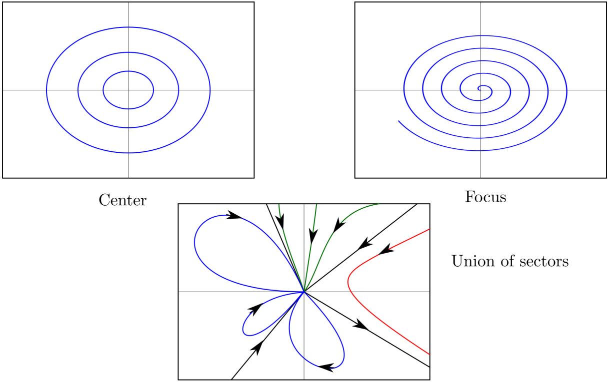
$$W^s_V = \{z = (x, y) \in U \ : \ \pi^x F^k(z) \in V, \ k \geq 0, \ F^k(z) \to 0\}$$

Dynamics of $x \mapsto x + p(x,0)$

The map $x \mapsto x + p(x,0)$ is the time one step of the Euler method of

$$\dot{x} = p(x,0).$$

The case $n = 2$ is well understood (for instance Argémi, 1968).



Center

Focus

Union of sectors

The study of the dynamics of $x \mapsto x + p(x)$, for $x \in \mathbb{C}^n$ is a subject of complex dynamics: Écalle, Hakim, Abate, ...

Hypotheses

For some norms, there exists $\rho_0 > 0$ such that

H1 There exists $a_p > 0$ such that

$$\|\mathrm{x} + p(x,0)\| \leq \|x\| - a_p \|x\|^N, \qquad x \in V_{\rho_0}.$$

H2 $D_x q(x,0) = 0$ and there exists $B_q > 0$ such that

$$\|\mathrm{Id} - D_y q(x,0)\| \leq 1 - B_q \|x\|^{M-1}, \qquad x \in V_{\rho_0}.$$

H3 There exists a constant $a_V > 0$ such that

$$\mathrm{dist}\big(x + p(x,0), \left(V_{\rho_0}\right)^{\mathrm{c}}\big) \geq a_V \|x\|^N, \qquad x \in V_{\rho_0}.$$

Main result. Existence and regularity

In the above setting, V star-shaped w.r.t.0, $F \in \mathcal{C}^r$, hypotheses H1, H2 and H3.
There exists $\ell_0 = \ell_0(p, q, V)$, s.t. if $r > \ell_0$ and there exist $K^{\leq}$, R of class C^{r^*} with

$$K^{\leq}(x) = (x, 0) + O_2, \qquad R(x) = x + p(x, 0) + O_{N+1}$$

and

$$F \circ K^{\leq} - K^{\leq} \circ R = O_\ell, \qquad \ell > \ell_0$$

then $\exists \rho$ and a unique $K^{>} : V_\rho \to U$, $K^{>} = O_{\ell-N+1}$ s.t. $K = K^{\leq} + K^{>}$ satisfies

$$F \circ K - K \circ R = 0$$

Regularity:
if cond (*) is satisfied $K \in C^{\min(r, r^*)}$
if cond (*) is not satisfied $K \in C^{r_0}$, r_0 depending on p, q, V.
Dependence on parameters

Computing an approximation

For this presentation we only consider the case $N = M$. We look for approximations

$$K^{\leq} = \sum_{j=1}^{\ell} K^j, \qquad R = \mathrm{Id} + \sum_{j=N}^{\ell+N-1} R^j, \qquad K^j, R^j \in \mathcal{H}^j,$$

such that

$$F \circ K^{\leq} - K^{\leq} \circ R \in \mathcal{H}^{>\ell+N-1}.$$

Computing $K^{\leq}$. First step

We proceed by induction. When $j = 1$,

$$K^1(x) = (x, 0)^\mathsf{T},$$

and

$$\begin{aligned} E^{>1} :&= F \circ K^1 - K^1 \circ R = K^1 + \begin{pmatrix} p(K^1) \\ q(K^1) \end{pmatrix} + \begin{pmatrix} f(K^1) \\ g(K^1) \end{pmatrix} - \begin{pmatrix} R \\ 0 \end{pmatrix} \\ &= \begin{pmatrix} x \\ 0 \end{pmatrix} + \begin{pmatrix} p(x,0) \\ q(x,0) \end{pmatrix} + \begin{pmatrix} f(x,0) \\ g(x,0) \end{pmatrix} - \begin{pmatrix} R(x) \\ 0 \end{pmatrix}. \end{aligned}$$

To obtain $E^{>1} \in \mathcal{H}^{>N}$ we take

$$R(x) = x + p(x, 0).$$

Computing $K^{\leq}$. Induction step

Assume that we have

$$K^{\leq j-1} = \sum_{i=1}^{j-1} K^i, \qquad R^{\leq N+j-2} = \mathrm{Id} + \sum_{j=N}^{N+j-2} R^i, \qquad K^i, R^i \in \mathcal{H}^i,$$

such that

$$E^{>j-1} := F \circ K^{\leq j-1} - K^{\leq j-1} \circ R^{\leq N+j-2} \in \mathcal{H}^{\geq N+j-1}.$$

We look for K^j, R^{j+N-1} such that

$$K^{\leq j} := K^{\leq j-1} + K^j, \qquad R^{\leq N+j-1} = R^{\leq N+j-2} + R^{N+j-1}$$

satisfy

$$E^{>j} := F \circ K^{\leq j} - K^{\leq j} \circ R^{\leq N+j-1} \in \mathcal{H}^{\geq N+j}.$$

Recall that $N = M$.

$$E^{>j} = F \circ \left(K^{\leq j-1} + K^j\right) - \left(K^{\leq j-1} + K^j\right) \circ \left(R^{\leq N+j-2} + R^{N+j-1}\right)$$
$$= \begin{pmatrix} E_x^{>j-1} \\ E_y^{>j-1} \end{pmatrix} + \begin{pmatrix} Dp(x,0)K^j - DK_x^j R^N - R^{N+j-1} \\ Dq(x,0)K^j - DK_y^j R^N \end{pmatrix} + \mathcal{O}(\|x\|^{N+j})$$

with $K^j = (K_x^j, K_y^j)$. We try to find K_x^j, K_y^j and R^{N+j-1} such that

$$D_x p(x,0)K_x^j + D_y p(x,0)K_y^j - DK_x^j\, p(x,0) - R^{N+j-1} = -E_x^{N+j-1}$$

and

$$D_y q(x,0)K_y^j - DK_y^j\, p(x,0) = -E_y^{N+j-1}.$$

Such conditions leads us to introduce the auxiliary equation

$$Dh(x)\mathbf{p}(x) - \mathbf{Q}(x)h(x) = \mathbf{w}(x)$$

Computing $K^{\leq}$. The auxiliary equation

Let

$$Dh(x)\mathbf{p}(x) - \mathbf{Q}(x)h(x) = \mathbf{w}(x).$$

Counting polynomial coefficients

$$x \in \mathbb{R}^n, \qquad h(x) \in \mathbb{R}^m, \qquad h \in \mathcal{H}^j, \qquad Dh \cdot \mathbf{p} - \mathbf{Q}h \in \mathcal{H}^{j+N-1}.$$

We have

- Number of coefficients

$$\binom{j+n-1}{n-1} m$$

- Number of equations

$$\binom{j+N-1+n-1}{n-1} m$$

This implies that the auxiliary equation generically has no polynomial solutions, except for $n = 1$.

Solutions of $Dh(x)\mathbf{p}(x) - \mathbf{Q}(x)h(x) = \mathbf{w}(x)$

We look for homogeneous solutions of

$$Dh(x)\mathbf{p}(x) - \mathbf{Q}(x)h(x) = \mathbf{w}(x).$$

Assume that $\mathbf{p} \in \mathcal{H}^N$, $\mathbf{Q} \in \mathcal{H}^{N-1}$ is a square matrix and $\mathbf{w} \in \mathcal{H}^{N+\nu}$ defined on the convex set V. In addition,

HP1 There exists $a_p > 0$ such that

$$\|\mathrm{x} + \mathbf{p}(x)\| \leq \|x\| - a_{\mathbf{p}}\|x\|^N, \qquad x \in V_{\rho_0}.$$

HP2 There exists a constant $a_V^{\mathbf{p}} > 0$ such that

$$\operatorname{dist}\left(x + \mathbf{p}(x), \left(V_{\rho_0}\right)^{\mathrm{c}}\right) \geq a_V^{\mathbf{p}}\|x\|^N, \qquad x \in V_{\rho_0}.$$

Notation

Some notation:

$$b_{\mathbf{p}} = \sup_{x \in V_{\rho_0}} \frac{\|\mathbf{p}(x)\|}{\|x\|^N}, \qquad B_{\mathbf{Q}} = -\sup_{x \in V_{\rho_0}} \frac{\|\mathrm{Id} - \mathbf{Q}(x)\| - 1}{\|x\|^{N-1}}$$

and

$$c_{\mathbf{p}} = a_{\mathbf{p}}, \ \text{ if } \ B_{\mathbf{Q}} \leq 0, \quad \text{and} \ \ c_{\mathbf{p}} = b_{\mathbf{p}} \ \text{ otherwise.}$$

Moreover, $\varphi(t, x)$ is the solution of

$$\dot{x} = p(x)$$

and $M(t, x)$ is the solution of

$$\dot{\psi} = \mathbf{Q}(\varphi(t, x))\psi, \qquad \psi(0) = \mathrm{Id}$$

The result

Let $\mathbf{p}, \mathbf{Q}, \mathbf{w}$ be $\mathcal{C}^r$. If $\nu + 1 + \frac{B_{\mathbf{Q}}}{c_{\mathbf{p}}} > \max\{1 - \frac{A_{\mathbf{p}}}{b_{\mathbf{p}}}, 0\}$, there exists a unique $\mathcal{C}^1$ solution $h \in \mathcal{H}^{\nu+1}$ of $\quad Dh(x)\mathbf{p}(x) - \mathbf{Q}(x)h(x) = \mathbf{w}(x) \quad$ given by:

$$h(x) = \int_{\infty}^{0} M^{-1}(t, x)\mathbf{w}(\varphi(t, x))\, dt, \qquad x \in V.$$

Concerning its regularity we have the following cases:

1. $A_{\mathbf{p}} > b_{\mathbf{p}}$. If $1 \le r \le \infty$, then h is $\mathcal{C}^r$ in V. If $\mathbf{p}, \mathbf{Q}, \mathbf{w}$ are real analytic functions in $\Omega(\gamma_0)$, h is analytic in $\Omega(\gamma_1)$ for some $\gamma_1 \le \gamma_0$.
2. $A_{\mathbf{p}} = b_{\mathbf{p}}$. If $1 \le r \le \infty$, then h is $\mathcal{C}^r$ in V.
3. $A_{\mathbf{p}} < b_{\mathbf{p}}$. Let $r_{\mathbf{p}}$ be the maximum of $1 \le i \le r$ such that

$$\nu + 1 + \frac{B_{\mathbf{Q}}}{c_{\mathbf{p}}} - i\left(1 - \frac{A_{\mathbf{p}}}{b_{\mathbf{p}}}\right) > 0.$$

   Then h is $\mathcal{C}^{r_{\mathbf{p}}}$ in V.

Loss of differentiability

Let $F = \varphi_{t=1}$ of

$$\dot{x} = p(x), \qquad \dot{y} = q_1(x)y + g(x), \quad x = (x_1, x_2) \in \mathbb{R}^2, y \in \mathbb{R}$$

and p is such that $\dot{x} = p(x)$ in polar coordinates is

$$\dot{r} = -ar^5, \qquad \dot{\theta} = r^4 \sin 4\theta$$

and

$$q_1(x) = b(x_1^2 + x_2^2)^2, \quad g(x) = 4(x_1^2 + x_2^2)x_1 x_2(x_1^2 - x_2^2).$$

Invariant manifold tangent to $y = 0$

$$h(x) = \int_{\infty}^{0} [M_y(t,x)]^{-1} \left[\varphi_r(t;r,\theta)\right]^6 \sin(4\varphi_\theta(t;r,\theta))\, dt$$

where φ is the solution of $\dot{x} = p(x)$ and
M_y is the solution of $\dot{M}_y = b(\varphi_r(t;r,\theta))^4 M_y$ with $M_y(0,r,\theta) = 1$.

$$h(x) = 4c_x \int_{\infty}^{0} \frac{1}{(1+4atr^4)^{\frac{b}{4a}+\frac{6}{4}-\frac{1}{a}} \cdot \left[c_x^2 + (1+4atr^4)^{\frac{2}{a}}\right]}\, dt$$

The regularity of h at points $(0, x_2)$ is the same as the regularity of

$$s^{2m-1} \log \frac{1}{s^2}$$

which is C^{2m-2}.

For differential equations the results are completely analogous. The homological equations depend on the same linear PDE.

Parabolic points and periodic orbits appear in Celestial Mechanics

- Moser (Sitnikov problem) 73
- McGehee 73
- Easton / Robinson 84
- Martínez-Piñol 94
- Martínez-Simó 14
- Guardia-Martín-Seara 15
- ...

An application: the spatial elliptic RTBP

Assume two point masses move under their mutual gravitational attraction force in elliptic motion. Let be μ and $1-\mu$ their masses, $0 \le \mu \le 1/2$. Their position $q_1, q_2 \in \mathbb{R}^3$ can be described by

$$q_1 = \mu \hat{q}, \quad q_2 = -(1-\mu)\hat{q}, \qquad \hat{q}(f) = (\rho(f)\cos f, \rho(f)\sin f, 0)$$

where $\rho(f)$, for a given eccentricity $0 \le e < 1$, satisfies

$$\rho(f) = \frac{1-e^2}{1+e\cos f}, \qquad \frac{df}{dt} = \frac{(1+e\cos f)^2}{(1-e^2)^{3/2}}.$$

A third massless body moves in $\mathbb{R}^3$ under the attraction of the other two. The position of the third body q satisfies

$$\ddot{q} = -(1-\mu)\frac{q-q_1}{r_1^3} - \mu\frac{q-q_2}{r_2^3}, \qquad r_i = \|q-q_i\|,\ i = 1,2.$$

This system is Hamiltonian with respect to

$$H(q,p,t) = \frac{\|p\|^2}{2} - U(q,t), \qquad U(q,t) = \frac{1-\mu}{r_1} + \frac{\mu}{r_2}. \qquad (1)$$

Preliminary changes:

- spherical coordinates (r, α, θ), (R, A, Θ)
- McGehee coordinates $r = 2/z^2$

The set $\{z = 0,\ R = 0\}$ is invariant and foliated by fixed points. We focus on those with $\theta = \Theta = 0$, $\alpha = \alpha_0$, $A = A_0$.
Further changes

- $$\hat{\theta} = \frac{\theta}{z}, \quad \hat{\Theta} = \frac{z\Theta}{\theta}, \quad \hat{\alpha} = \frac{\alpha - \alpha_0 + AR}{z}, \quad \hat{A} = \frac{A - A_0}{z},$$
- $$u = (z + R)/2, \quad v = (z - R)/2.$$

Let X denote the vector after the changes of variables. It can be written in the form of our theorem taking $x = (u, \hat{\Theta})$, $y = (v, \hat{\alpha}, \hat{A}, \hat{\theta})$ with

$$p(x,y) = \begin{pmatrix} -\frac{1}{4}(u+v)^3 u \\ -\frac{1}{4}(u+v)^2(u-v)\hat{\Theta} \end{pmatrix}, \; q(x,y) = \begin{pmatrix} \frac{1}{4}(u+v)^3 v \\ \frac{1}{4}(u+v)^2(u-v)\hat{\alpha} \\ \frac{1}{4}(u+v)^2(u-v)\hat{A} \\ \frac{1}{4}(u+v)^2(u-v)\hat{\theta} \end{pmatrix}.$$

which satisfies the hypotheses of our theorem. Hence, the origin of this system (which corresponds to any point in the manifold at infinity with $\theta = \Theta = 0$) has parabolic two dimensional attracting manifolds. Our theorem states that they can be approximated by sums of homogeneous functions of increasing degree.
The application of our method provides the following unexpected result.

The parabolic manifolds admit polynomial approximation

Theorem

The origin of the previous system is parabolic. It has a stable parabolic invariant manifold that admits polynomial expansion up to any order, that is, for any $\ell \geq 1$ there exist $K(x,t)$, 2π-periodic in t, and $Y(x)$ of the form

$$K(x,t) = (x,0) + \sum_{l=2}^{\ell} \check{K}^l(x,t), \quad Y(x) = p(x,0) + \sum_{l=5}^{7} Y^l(x),$$

where $\check{K}^l$ and Y^l are homogeneous polynomials of degree l, such that

$$X(K(x,t),t) - DK(x,t)Y(x) - \partial_t K(x,t) = o(\|x\|^{\ell}).$$

Gevrey character of 1D manifolds

We consider $(\bar{x}, \bar{y}, \bar{z}) = F(x, y, z)$, $x \in \mathbb{R}$, $y \in \mathbb{R}^m$, $z \in \mathbb{R}^{m'}$, with

$$\begin{aligned}\bar{x} &= x - ax^N + x^{N-1}\langle v, y\rangle + x^{N-1}\langle w, z\rangle + \tilde{f}_N + f_{\geq N+1},\\ \bar{y} &= y + x^{M-1}B_1 y + x^{M-1}B_2 z + \tilde{g}_M + g_{\geq M+1},\\ \bar{z} &= Cz + h_{\geq 2}(x, y, z)\end{aligned}$$

- $N, M \geq 2$, $\quad a \neq 0$
- $0 \notin \operatorname{Spec} B_1$ if $M < N$, $-aj \notin \operatorname{Spec} B_1$, $\forall j \in \mathbb{N}$, if $M = N$
- $1 \notin \operatorname{Spec} C$

I. Baldomá-À. Haro. One dimensional invariant manifolds of Gevrey type in real-analyitc maps. DCDS, 295–322, 2008.

We look for the formal solution

$$K(t) = \sum K_j t^j, \qquad R(t) = t - at^N + bt^{2N-1}$$

of $F \circ K = K \circ R$. We want to obtain bounds of Gevrey type:

$$|K_j| \leq C\, D^j j!^{\gamma}$$

The series are Gevrey $\gamma = \alpha$ with $\alpha = \frac{1}{N-1}$ if $N \leq M$ and $\alpha = \frac{1}{N-M}$ if $M < N$

Formal computation of the manifold

Notation:

$$\mathcal{L}(x,y,z) = \begin{pmatrix} x - ax^N \\ y \\ Cz \end{pmatrix}, \qquad G(x,y,z) = F(x,y,z) - \mathcal{L}(x,y,z), \tag{2}$$

and the family of operators ($l \geq 2$)

$$\mathcal{A}_l = \begin{cases} -(B_1 + al\,\mathrm{Id}\,)^{-1}, & \text{if} \quad N = M, \\ -B_1^{-1}, & \text{if} \quad M < N, \\ -(la)^{-1}\mathrm{Id}\,, & \text{if} \quad M > N \end{cases} \tag{3}$$

and

$$L = \min\{N, M\}.$$

We want to solve $F \circ K = K \circ R$.

$$K_l^z = -(C - \mathrm{Id}\,)^{-1} E_l^z,$$

$$K_l^y = \begin{cases} \mathcal{A}_l E_{l+L-1}^y, & \text{if} \quad N < M, \\ \mathcal{A}_l (E_{l+L-1}^y + \tilde{B} K_l^z), & \text{if} \quad N \geq M, \end{cases}$$

$$K_l^x = \begin{cases} \dfrac{-1}{a(l-N)} (E_{l+N-1}^x + v^\top K_l^y + w^\top K_l^z), & \text{if} \quad l \neq N, \\ c, & \text{if} \quad l = N, \end{cases}$$

and

$$R_{l+N-1} = \begin{cases} 0, & \text{if} \quad l > 1, \quad l \neq N, \\ b = E_{2N-1}^x + v^\top K_N^y + w^\top K_l^z, & \text{if} \quad l = N; \end{cases}$$

$$E^x_{l+N-1} = -\,a \sum_{\substack{l_1+\cdots+l_N=l+N-1 \\ 1\le l_i\le l-1}} \prod_{i=1}^{N} K^x_{l_i} + \sum_{k=N}^{l+N-1} \sum_{\substack{l_1+\cdots+l_k=l+N-1 \\ 1\le l_i\le l-1}} G^x_k[K_{l_1},\cdots,K_{l_k}]$$
$$- \sum_{k=2}^{l-1} K^x_k \sum_{\substack{l_1+\cdots+l_k=l+N-1 \\ 1\le l_i\le l+N-2}} \prod_{i=1}^{k} R_{l_i}, \tag{4}$$

$$E^y_{l+L-1} = \sum_{k=M}^{l+L-1} \sum_{\substack{l_1+\cdots+l_k=l+L-1 \\ 1\le l_i\le \min\{l-1,l+L-M\}}} G^y_k[K_{l_1},\cdots,K_{l_k}]$$
$$- \sum_{k=M-L+2}^{\min\{l-1,l+L-N\}} K^y_k \sum_{\substack{l_1+\cdots+l_k=l+L-1 \\ 1\le l_i\le l+N-2}} \prod_{i=1}^{k} R_{l_i} \tag{5}$$

and

$$E^z_l = \sum_{k=2}^{l} \sum_{\substack{l_1+\cdots+l_k=l \\ 1\le l_i\le l-1}} G^z_k[K_{l_1},\cdots,K_{l_k}] - \sum_{k=2}^{l-N+1} K^z_k \sum_{\substack{l_1+\cdots+l_k=l \\ 1\le l_i\le l+N-2}} \prod_{i=1}^{k} R_{l_i}. \tag{6}$$

We perform some preliminary changes and a rescaling of size λ. Those changes do not afect the possible Gevrey character of the parameterization.

Using an induction procedure we get that there exists $\lambda > 1$ big enough such that the coeffients of K corresponding to the rescaled map satisfy

$$\|K_j\| \leq Cj!^{\alpha}$$

Gevrey functions and Gevrey series

Maurice Gevrey in "Sur la nature analytique des solutions des équations aux dérivées partielles"

Ann. Scien. E.N.S. (1918)

$\varphi \in C^\infty(a,b)$ is of class α in (a,b) if $\exists M, R > 0$ s.t.

$$\left|\frac{d^n\varphi}{dx^n}(x)\right| < MR^{-n}\Gamma(\alpha n) \sim \left(\alpha^{1/2}(2\pi)^{(1-\alpha)/2}M\right)(\alpha R^{-1})^n n!^\alpha, \quad x \in (a,b),$$

If $\alpha = 1$ then φ is analytic with a radius of convergence bigger or equal than R/α. If $\alpha < 1$ then φ is entire.

Gevrey series

Balser: a formal series $\sum a_n z^n$ is Gevrey of order $1/k$ if $\exists C, K$ s.t.

$$|a_n| \leq CK^n\Gamma(1+\frac{n}{k}), \qquad n \geq 0$$

M. Canalis-Durand: a formal series $\sum a_n z^n$ is Gevrey of order $1/k$ and type A if $\exists C, \alpha$ s.t.

$$|a_n| \leq CA^{n/k}\Gamma(\alpha+\frac{n}{k}), \qquad n \geq 0$$

Let $\mathbb{C}[[z]]_{1/k,A}$ be the set of formal series of order $1/k$ and type A

Gevrey functions are particular cases of ultradifferentiable functions: C^∞ functions whose derivatives satisfy growth restrictions on their derivatives.

Concretely, given a sequence $M = (M_n)$ of positive numbers, we say that $f : \Omega \subset \mathbb{R}^p \to \mathbb{R}^q$, $f \in C^\infty$ is of class M ($f \in \mathcal{E}^M$) if for any $x \in \Omega$, $\exists$ a nbh U and numbers $A, B > 0$ s.t.

$$\frac{1}{n!}\|D^n f(x)\| \leq AB^n M_n, \qquad x \in U, n \geq 0.$$

(Denjoy Carleman classes of Roumieu type)
Gevrey classes of order α are obtained when $M_n = n!^\alpha$.
Fact If $\exists \lambda \geq 1$ s.t.

$$\sup_{k_1+\cdots+k_r=k} M_r M_{k_1} \dots M_{k_r} \leq \lambda^k M_k$$

then the class $\mathcal{E}^M$ is closed under composition.

Now we concentrate on functions defined on sectors

$$S = \{z \mid 0 < |z| < r, \beta_1 < \arg z < \beta_2\}$$

We say that $f : S \to \mathbb{C}$ is (Poincaré) asymptotic to a formal series $\sum a_n z^n$ in S when $z \mapsto 0$ if for all S_1, $\overline{S}_1 \subset S \cup \{0\}$, $N \geq 0$, $\exists C_{S_1,N}$ s.t.

$$|f(z) - \sum_{n=0}^{N-1} a_n z^n| \leq C_{S_1,N} |z|^N, \qquad z \in S_1$$

We say that $f : S \to \mathbb{C}$ is (Gevrey) asymptotic to a formal series $\sum a_n z^n$ in S when $z \mapsto 0$ if it is Poincaré asymptotic and $C_{S_1,N} \leq C_{S_1} A^N N!^\alpha$

Let $\mathcal{A}_{\alpha,A}(S)$ be the set of analytic functions on S that are Gevrey asymptotic to some formal series. Let $\mathcal{A}_\alpha(S) = \cup_{A>0} \mathcal{A}_{\alpha,A}(S)$

Property If $f \in \mathcal{A}_{\alpha,A}(S)$ then the formal series $\sum a_n z^n$ is Gevrey

One can consider the maps

$$J_1 : \mathcal{A}(S) \to \mathbb{C}[[z]]$$
$$J_2 : \mathcal{A}_{\alpha,A}(S) \to \mathbb{C}[[z]]_{\alpha,A}$$
$$J_3 : \mathcal{A}_{\alpha}(S) \to \mathbb{C}[[z]]_{\alpha}$$

They are morphisms of differential algebras.

Theorem of Borel-Ritt-Gevrey
If opening of S is less than $\alpha\pi$ then J_3 is exhaustive.

Once we know a Gevrey formal series K solution of $F \circ K = K \circ R$ with R as before we can use the previous tools to get an analytic solution K on some sector.

Let $\beta < \alpha\pi$. We have that

- $\exists \rho > 0$ and $K_B : S(\beta, \rho) \to U \subset \mathbb{C}^{1+m+m'}$ analytic Gevrey asymptotic to K of order α.
- $\exists c, \kappa > 0$ s.t.

$$\|F \circ K_B(t) - K_B \circ R(t)\| \leq c \exp(-\kappa |t|^{-1/\alpha}), \qquad t \in S(\beta, \rho)$$

We look for Δ s.t. $K_B + \Delta$ is solution of the invariance equation.

Working with the equation we arrive to an equivalent fixed point equation

$$\Delta = -\mathcal{L}_0^{-1}[\mathcal{N} \circ (K_B + \Delta) - \mathcal{N} \circ K_B - \Delta \circ R + E]$$

where $\mathcal{L}_0$ is the linear part of F and $\mathcal{N} = F - \mathcal{L}_0$ and

$$E = F \circ K_B - K_B \circ R$$

We work in the space

$$\Sigma_{\kappa,\alpha}(S) = \{\Delta : \overline{S} \to \mathbb{C}^{1+m+m'} \mid \text{ continuous, analytic on } S, \\ \|\Delta\|_\Sigma := \sup \exp(\kappa |t|^{-1/\alpha}) \|\Delta(t)\| < \infty\}$$

In our case we can apply this strategy when $N \geq M$ to get an analytic K.

Not all manifolds are Gevrey

Let

$$F\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} x - ax^N + vx^{N-1}y + \tilde{f}_N + f_{\geq N+1} \\ y + bx^{N-1}y + \tilde{g}_N + g_{\geq N+1} \end{pmatrix}$$

If $a, b > 0$, F has a parabolic curve tangent to the x-axis which is the graph of a Gevrey function φ_F.

For any $p \geq 1$ there exists H s.t.

$$F(x, y) - H(x, y) = O(\|(x, y)\|^{N+p})$$

and H has a parabolic curve which is the graph of a polynomial φ_H. Moreover

$$\varphi_F(x) - \varphi_H(x) = O(\|x\|^{p+1})$$

Examples of optimal Gevrey order for some formal solutions

Infinity manifolds of the Sitnikov problem

Martínez-Simó have shown they are Gevrey of order 1/3. If this case $N = M = 4$ so that $\alpha = 1/(N-1)$.

R. Martínez-C. Simó. On the regularity of the infinity manifolds: the case of Sitnikov problem and some global aspects of the dynamics. Talk at Fields Institute, Toronto, 2009.

Maps with $M < N$

$$\begin{cases} \dot{x} = & -ax^N \\ \dot{y} = & bx^{M-1}y + cx^{M+1}, \end{cases} \qquad M \geq 2, \quad N \geq M+1$$

Let

$$F\begin{pmatrix} x \\ y \end{pmatrix} = \varphi_{t=1}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} x \\ y \end{pmatrix} + \begin{pmatrix} -ax^N \\ bx^{M-1}y + cx^{M+1} \end{pmatrix}$$

$$+ \begin{pmatrix} -a^2Nx^{2N-1} \\ -ab(M-1)x^{M+N-2}y + b^2x^{2M-2}y - ac(M+1)x^{M+N} + bcx^{2M} \end{pmatrix}$$

$$+ \dots$$

Invariance eq. for graph h

$$-ax^N h'(x) = bx^{M-1} h(x) + cx^{M+1}$$

We look for $h(x) = \sum_{n\geq 2} h_n x^n$

$$\begin{aligned} h_n &= \tfrac{-c}{b}(\tfrac{-a}{b})^i (N-M)^i \frac{\Gamma(\frac{2}{N-M}+i)}{\Gamma(\frac{2}{N-M})}, && \text{if } n = 2 + i(N-M) \\ h_n &= 0, && \text{otherwise} \end{aligned}$$

and hence

$$h_n = CD^n \Gamma(\alpha n), \qquad D = \left(\frac{-a}{b\alpha}\right)^{\alpha}, \qquad \alpha = \frac{1}{N-M}$$

with $\Gamma(\alpha n) \sim n!^{\alpha}$